Asymptotic behaviour of the autocorrelation function of continuous time moving average processes

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Joint work with Serge Cohen, Toulouse

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Fractional Lévy processes

$L = (L_t)_{t \geq 0}$ Lévy process

$$E(L_1) = 0, \quad \mathrm{Var}(L_1) < \infty, \quad d \in (0, 1/2)$$

$$M_d(t) := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[(t-s)_+^d - (-s)_+^d \right] dL_s$$

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Goal: Estimate d (resp. H), based on observations $X_1, X_2, \ldots, X_n$.

If B Brownian motion, then M_d selfsimilar, and many estimators for d exist based on this property.
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$$\begin{aligned} \gamma_X(h) &= \frac{\sigma_0^2}{2}\left(|h+1|^{2H} - 2|h|^{2H} + |h-1|^{2H}\right) \\ & \quad \sigma_0^2 = \mathrm{Var}(X_1) \\ \rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)} \end{aligned}$$

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Idea: Estimate $\gamma_X(h)$, $h = 1, \ldots, \overline{h}$, and get H (hence $d = H - 1/2$) using a moment estimator. [Alternatively, could try spectral density estimator.]

$$
\begin{aligned}
\gamma_X(0) &= \sigma_0^2 \\
\gamma_X(1) &= \frac{\sigma_0^2}{2}(2^{2H} - 2) \\
\Longrightarrow \rho_X(1) &= 2^{2H-1} - 1 \\
\Longrightarrow H &= \frac{1}{2}\left(1 + \frac{\log(\rho_X(1) + 1)}{\log 2}\right)
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Hence if

$$\hat{\gamma}_X(h) := \frac{1}{n}\sum_{k=1}^{n} X_k X_{k+h}$$

estimator for $\gamma_X(h)$, then

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Question: Asymptotic properties of $\hat{\gamma}_X(0), \ldots, \hat{\gamma}_X(\bar{h}), \hat{H}$?

Properties of the estimator

$(X_t)_{t\in\mathbb{Z}}$ is mixing in the ergodic theoretic sense, i.e.

$$\lim_{t\to\infty} P(X_0 \in A,\, X_t \in B) = P(X_0 \in A)\, P(X_0 \in B) \quad \forall\, A, B \in \mathcal{B}(\mathbb{R})$$

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Asymptotic normality? Usually proved showing strong mixing conditions, but fractional Lévy process is not strongly mixing. Hence need other concepts.

$$\begin{aligned} X_t &= M_d(t+1) - M_d(t) \\ &= \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[[t+1-s)_+^d - (-s)_+^d \right] - \left[(t-s)_+^d - (-s)_+^d \right] dL_s \\ &= \int_{-\infty}^{\infty} f(t-s)\, dL_s, \end{aligned}$$

with

$$f(t) := (t+1)_+^d - t_+^d.$$

(Here: $f \notin L^1$, $f \in L^2$.)

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Theory for asymptotic behaviour of ACF of Lévy driven continuous time moving average processes is needed.

Setup

Continuous time MA:

$L = (L_t)_{t \geq 0}$ Lévy process

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$$f : \mathbb{R} \to \mathbb{R}, \quad f \in L^2$$

$$X_t = \mu + \int_{-\infty}^{\infty} f(t-s)\, dL_s \tag{1}$$

Discrete time MA:

$(Z_t)_{t \in \mathbb{Z}}$ i.i.d.

$E(Z_0) = 0, \quad \mathrm{Var}(Z_1) = \sigma_Z^2 < \infty$

$$\sum_{j \in \mathbb{Z}} |\psi_j| < \infty$$

$$Y_t = \mu + \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j} = \mu + \sum_{j \in \mathbb{Z}} \psi_{t-j} Z_j \tag{2}$$

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$$\bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} Y_k \sim AN(\mu, \frac{1}{n} v), \quad v = \sigma_Z^2 (\sum_{j=-\infty}^{\infty} \psi_j)^2 = \sum_{h=-\infty}^{\infty} \gamma_Y(h).$$

Theorem: [Continuous time MA, sample mean]

$$X_t = \mu + \int_{-\infty}^{\infty} f(t-s)\, dL_s$$

$$\text{Let} \quad F : [0,1] \to [0,\infty], \quad F(u) := \sum_{j=-\infty}^{\infty} |f(u+j)|$$

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Then

$$v := \sum_{h=-\infty}^{\infty} \gamma(h) = \sigma_L^2 \int_0^1 \left(\sum_{j=-\infty}^{\infty} f(u+j) \right)^2 du \overset{\text{if } f \geq 0}{=} \sigma_L^2 \int_0^1 F^2(u)\, du$$

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converges and

$$\bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \sim AN(\mu, \frac{1}{n} v)$$

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- Idea of proof: First for f with compact support (m-dependent sequences) and then apply a variant of Slutsky's lemma, as in discrete time.

Bartlett's formula

Suppose $Y_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, $\sum |\psi_j| < \infty$, (Z_t) i.i.d. noise with

$$E(Z_0^4) =: \eta\,(\sigma_Z^2)^2 < \infty$$

$$\hat{\gamma}_n(h) := \frac{1}{n}\sum_{t=1}^{n} X_t X_{t+h}, \quad \hat{\rho}_n(h) := \frac{\hat{\gamma}_n(h)}{\hat{\gamma}_n(0)}.$$

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Then

$$(\hat{\rho}_n(1), \ldots, \hat{\rho}_n(h)) \sim AN((\rho(1), \ldots, \rho(h)), n^{-1}W), \quad W = (w_{ij})_{i,j=1,\ldots,h}$$

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$$\begin{aligned} w_{ij} \;=\; \sum_{k=-\infty}^{\infty} \big[& \rho(k+i)\rho(k+j) + \rho(k-i)\rho(k+j) + 2\rho(i)\rho(j)\rho^2(k) \\ & -2\rho(i)\rho(k)\rho(k+j) - 2\rho(j)\rho(k)\rho(k+i) \big] \end{aligned}$$

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w_{ij} does not depend on η. Same result also without fourth moment assumption but quicker decrease of ψ_j

Theorem: [Sample autocorrelation]

L Lévy process such that

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Denote

$$G : [0,1] \to [0,\infty], \quad G(u) := \sum_{j=-\infty}^{\infty} f(u+j)^2$$

and suppose

$$G \in L^2([0,1]) \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \Bigg(\underbrace{\int_{-\infty}^{\infty} |f(s)f(s+k)|\, ds}_{=\sigma_L^{-2}\gamma(k),\ if\ \ f\geq 0} \Bigg)^2 < \infty$$

(then $f \in L^4(\mathbb{R})$). Denote for $q \in \mathbb{Z}$

$$g_q : [0,1] \to \mathbb{R}, \quad u \mapsto \sum_{j=-\infty}^{\infty} f(u+j)f(u+j+q)$$

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$$\tilde{w}_{ij} = \underbrace{w_{ij}}_{Bartlett} + \underbrace{\frac{(\eta - 3)\sigma_L^4}{\gamma(0)^2} \int_0^1 \left[g_i(u) - \rho(i) g_0(u) \right] \left[g_j(u) - \rho(j) g_0(u) \right] du}_{=: \tilde{w}_{ij}^{(2)}}$$

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- If $f(x) = \mathbf{1}_{[0,1/2)} + \mathbf{1}_{[1,2)}$ and $\eta \neq 3$, then $\tilde{w}_{11}^{(2)} \neq 0$ (corresponds to sampling discrete time MA process only at even times)

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- Results connected to results of Peccati, Taqqu and coauthors.