Problems concerning mixed SDEs: The rate of convergence of Euler approximations of solution of mixed SDE and financial applications of mixed models

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Outline

Introduction

1. We study the Euler approximations of the solution of mixed stochastic differential equation driven simultaneously by the fractional Brownian motion with the Hurst parameter $H > 1/2$ and the Wiener process. The error of the approximations is estimated via the norm in some Besov spaces.

Several objects processed in time have a component with a long memory (that is modeled by fBm with $H \in (1/2, 1)$) and also a component without memory (that is modeled by a Wiener process). Therefore it is natural to consider stochastic differential equation involving both standard Brownian motion (Wiener process) and fractional Brownian motion.

Numerical solution via the time discretization of SDEs driven by Brownian motion has the long history. We refer to the monograph of Kloeden, Platen, containing the theory of numerical solution of such SDEs with regular coefficients.
As to Euler approximations for SDEs driven by fBm, we mention the paper of Nourdin and Neuenkirch that studies Euler approximations for homogeneous one-dimensional SDEs with bounded coefficients having bounded derivatives up to the third order. The paper of M., Shevchenko focuses on discrete-type approximations of solutions to non-homogeneous stochastic differential equations involving fractional Brownian motion. Nowadays the SDEs with more irregular fBm ($H < 1/2$) (it is rough path theory) are treated. In a general context, strong and weak approximations to Gaussian rough paths have been studied by Friz and Victoir, Lejay, Neuenkirch, Nourdin, Tindel. Approach to expansion of the functionals from the solution of SDE w.r.t. fBm, and the error estimate was considered by Neuenkirch, Nourdin, Rössler, Tindel. As to the mixed stochastic differential equations, the existence and uniqueness of the solution in the case $H \in (3/4, 1)$ is contained in M., Posashkov.

Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space, with a filtration satisfying standard conditions. Denote $\{W_t, \mathcal{F}_t,\ t \in [0,T]\}$ the standard Brownian motion, adapted to this filtration.

Definition 1. The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B_t^H = \{B_t^H, \mathcal{F}_t,\ t \geq 0\}$, having the properties $B_0^H = 0,\ \mathrm{E}B_t^H = 0$, and
$\mathrm{E}B_t^H B_s^H = 1/2(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$.

Remark 1. Fractional Brownian motion has a continuous modification, according to the Kolmogorov theorem. In what follows we consider this continuous modification. Also, we suppose that our fBm is adapted to the filtration $\mathcal{F}_t, t \in [0,T]$. (We can suppose that $\mathcal{F}_t, t \in [0,T]$ is generated by W and B^H.)

Remark 2. Evidently, for $H = 1/2$ fBm is a standard Brownian motion. For $H \neq 1/2$ fBm has so called "memory" property. For $H \in (1/2, 1)$, B^H has "long memory" property. Further we consider only fBm with Hurst index $H \in (1/2, 1)$.

In order to introduce the pathwise integrals w.r.t. fBm consider two nonrandom functions f and g, defined on some interval $[a,b] \subset R$. Suppose also that the limits
$f(u+) := \lim_{\delta \downarrow 0} f(u+\delta)$ and $g(u-) := \lim_{\delta \downarrow 0} g(u-\delta),\ a \le u \le b$ exist.
Put
$f_{a+}(x) := (f(x) - f(a+))1_{(a,b)}(x),\ g_{b-}(x) := (g(b-) - g(x))1_{(a,b)}(x)$.
Suppose that $f_{a+} \in I^{\alpha}_{a+}(L_p[a,b])),\ g_{b-} \in I^{1-\alpha}_{b-}(L_q[a,b]))$, for some $p \ge 1,\ q \ge 1, 1/p + 1/q \le 1,\ 0 \le \alpha \le 1$. Consider fractional derivatives

$$(D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(1-\alpha)}\Big(\frac{f(s)}{(s-a)^{\alpha}} + \alpha \int_{\alpha}^{s} \frac{f(s)-f(u)}{(s-u)^{1+\alpha}} du\Big) 1_{(a,b)}(x),$$

$$(D^{1-\alpha}_{b-} g_{b-})(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\Big(\frac{g_{b-}(s)}{(b-s)^{1-\alpha}} + (1-\alpha)\int_{s}^{b} \frac{g_{b-}(s)-g_{b-}(u)}{(s-u)^{2-\alpha}} du\Big)$$
$$\times 1_{(a,b)}(x).$$

It is known that $D^{\alpha}_{a+} f_{a+} \in L_p[a,b],\ D^{1-\alpha}_{b-} g_{b-} \in L_q[a,b]$.

Definition 2. (Zaehle, Nualart, Rascanu) Under above assumptions, the generalized (fractional) Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx + f(a+)(g(b-) - g(a+)),$$

and for $\alpha p < 1$ it can be simplified to

$$\int_a^b f(x)dg(x) := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx.$$

As it follows from SKM, for any $1 - H < \alpha < 1$ there exists fractional derivative $D_{b-}^{1-\alpha} B_{b-}^H(x) \in L_\infty[a, b]$. Therefore, for $f \in I_{a+}^{\alpha}(L_1[a, b])$ we can define the integral w.r.t. fBm in the following way:

Definition 3. (Zaehle, Nualart, Rascanu) The integral with respect to fBm is defined as

$$\int_a^b f dB^H := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} B_{b-}^H)(x)dx.$$

Consider the following functional spaces. For $0 < \theta < 1$ let $W_1^\theta = W_1^\theta[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$:

$$\|f\|_{1,\theta} := \sup_{0 \le s < t \le T} \Big(\frac{|f(t) - f(s)|}{(t-s)^\theta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\theta}} du \Big) < \infty.$$

also, let $W_2^\theta = W_2^\theta[0, T]$ be the space of real-valued measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$\|f\|_{2,\theta} := \int_0^T \frac{|f(s)|}{s^\theta} ds + \int_0^T \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{1+\theta}} duds < \infty.$$

Note that the space W_2^θ is Banach space with respect to corresponding norms and $\|f\|_{1,\theta}$ is a seminorm. Of course, we can extend the definition of the space $W_2^\theta[0, T]$ to any subinterval $[s, t] \subset [0, T]$, and consider the space $W_2^\theta[s, t]$ with the norm

$$\|f\|_{2,\theta,s,t} := \int_s^t \frac{|f(u)|}{(u-s)^\theta} du + \int_s^t \int_0^v \frac{|f(v) - f(r)|}{(v-r)^{1+\theta}} drdv.$$

Some results concerning mixed stochastic differential equations

Consider one-dimensional stochastic differential equation of the form

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(s, X_s)dB_s^H, \ t \in [0, T], \quad (1)$$

where X_0 is $\mathcal{F}_0$-measurable random variable with finite moments of any order, W is standard Brownian motion, B^H is fBm with Hurst index $H \in (\frac{1}{2}, 1)$, the first integral in the right-hand side is Lebesgue-Stieltjes integral, the second one is stochastic integral with respect to standard Brownian motion, and the third one is the generalized Lebesgue-Stieltjes integral from the Definition 3.

We suppose that the coefficients satisfy the following assumptions

(A) There exists such $K > 0$ that for any $s \in [0, T]$ and $x \in \mathbb{R}$ $|a(s,x)| + |b(s,x)| + |c(s,x)| \leq K$.

(B) There exists such $L > 0$ that for any $t \in [0, T]$ and $x, y \in \mathbb{R}$

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| + |c(t,x) - c(t,y)| \leq L|x-y|.$$

(C) The function $c(t,x)$ is differentiable in x and there exist such constant $B > 0$ and parameter $\beta \in (1-H, 1)$ that for any $s, t \in [0, T]$ and $x \in \mathbb{R}$

$$|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| + |c(s,x) - c(t,x)| + |\partial_x c(s,x) - \partial_x c(t,x)| \leq B|s-t|^{\beta}.$$

(D) Partial derivative $\partial_x c(t,x)$ is Hölder continuous in x : there exist such constant $D > 0$ and parameter $\rho \in (3/2 - H, 1)$ that for any $t \in [0, T]$ and $x, y \in R$

$$|\partial_x c(t,x) - \partial_x c(t,y)| \leq D|x-y|^{\rho}.$$

As it was stated in [M.,Posashkov], under the conditions (A)–(D) the equation (1), considered for the values of Hurst index $H \in (\frac{3}{4}, 1)$, has the unique solution $\{X_t, t \in [0, T]\}$, and this solution belongs to Besov space $W_3^\gamma([0, T])$, $0 < \gamma < \min(1/2, \beta, \rho/2, \rho - 1/2)$. Here

$$W_3^\gamma([0, T]) := \{Y = Y_t(\omega) : (t, \omega) \in [0, T] \times \Omega, ||Y||_\gamma < \infty\},$$

with the norm

$$\|Y\|_\gamma^2 := \sup_{t \in [0,T]} \left(\mathrm{E} Y_t^2 + \mathrm{E} \left(\int_0^t \frac{|Y_t - Y_s|}{(t-s)^{1+\gamma}} ds \right)^2 \right). \qquad (2)$$

However, as we shall see from the calculations below, the rate of convergence will be the same for any $H \in (\frac{1}{2}, 1)$ provided that the solution of the equation (1) exists, is unique and the conditions (A)–(D) hold. Therefore, in what follows it will be our case.

Preliminary properties of Euler approximations

Now, let $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$, $\delta = T/n$, $\tau_k = k\delta$. Consider discrete Euler approximation of solution of equation (1):

$$X^\delta_{\tau_{k+1}} = X^\delta_{\tau_k} + a(\tau_k, X^\delta_{\tau_k})(\tau_{k+1} - \tau_k) + b(\tau_k, X^\delta_{\tau_k})(W_{\tau_{k+1}} - W_{\tau_k}) + c(\tau_k, X^\delta_{\tau_k})(B^H_{\tau_{k+1}} - B^H_{\tau_k}),$$

where $X^\delta_{\tau_0} = X_0$.
If $t_u = \max\{\tau_n : \tau_n \le u\}$, then the corresponding continuous interpolation satisfies the equation

$$X^\delta_u = X^\delta_{t_u} + a(t_u, X^\delta_{t_u})(u - t_u) + b(t_u, X^\delta_{t_u})(W_u - W_{t_u}) + c(t_u, X^\delta_{t_u})(B^H_u - B^H_{t_u}),$$

or, in the integral form,

$$X^\delta_u = X^\delta_0 + \int_0^u a(t_s, X^\delta_{t_s})ds + \int_0^u b(t_s, X^\delta_{t_s})dW_s + \int_0^u c(t_s, X^\delta_{t_s})dB^H_s. \quad (3)$$

At first we establish growth and Hölder property of trajectories of Euler approximations $\{X_t^\delta, t \in [0, T]\}$. In what follows we denote C the constants which values will be not so important.

Theorem 1. *Let the conditions (A)–(D) hold. Then the following statements are true.*

1) There exists such random variable $C(\omega)$ *having moments of any order that* $|X_t^\delta| \le C(\omega),\ t \in [0, T]$;

2) For any $0 < \eta < 1/2$ *there exists such random variable* $C(\omega, \eta)$ *having moments of any order that* $|X_t^\delta - X_s^\delta| \le C(\omega, \eta)|t - s|^{1/2-\eta},\ s, t \in [0, T]$.

Remark 1. Corresponding Hölder properties of the solution of the equation (1) can be established similarly.

Theorem 2. *Under the conditions (A)–(D) for any* $0 < \gamma < \min(1/2, \beta, \rho/2, \rho - 1/2)$ *there exists a constant* $C = C(\gamma)$, *depending on* γ *but not on* δ, *such that* $\|X^\delta\|_\gamma^2 \le C(\gamma)$ *for any* $\delta > 0$.

The rate of convergence in Besov space of Euler approximations

Introduce the notations

$$C_1(\omega, \eta, s, t) := C_\eta \cdot \left(\int_s^t \int_s^t \frac{|W_y - W_x|^{2/\eta}}{|x-y|^{1/\eta}} dx dy \right)^{\eta/2}, \qquad (4)$$

where C_η is some constant,

$$C_2(\omega, \alpha, s, t) = \sup_{s \le u \le v \le t} |(D_{v-}^{1-\alpha} B_{v-}^H)(u)|. \qquad (5)$$

Now we fix some $\alpha \in (1 - H, 1/2)$, sufficiently small $\eta > 0$, and consider the values of $C_1(\omega, \eta, 0, t)$ and $C_2(\omega, \alpha, 0, t)$ as the stochastic processes with respect to $t \in [0, T]$. Evidently, they are increasing and continuous in t. Further, for any $t \in [0, T]$ both the random variables $C_1(\omega, \eta, 0, t)$ and $C_2(\omega, \alpha, 0, t)$ have the moments of any order. So, if we define the stopping times $\tau_N := \inf\{t \in [0, T] : C_1(\omega, \eta, 0, t) + C_2(\omega, \alpha, 0, t) > N\} \wedge T$ then $C_1(\omega, \eta, 0, t \wedge \tau_N) \le N$, $C_2(\omega, \alpha, 0, t \wedge \tau_N) \le N$, and for a.s. $\omega \in \Omega$

we have that $\tau_N = T$ for any $N > N(\omega)$. For any $N > 0$ define $t(N) = t \wedge \tau_N$ and the norm

$$||X^\delta - X||^2_{\gamma,N} := \sup_{0\le t\le T} \mathsf{E}(X_{t(N)} - X^\delta_{t(N)})^2$$
$$+\mathsf{E}\left(\int_0^{t(N)} |X_{t(N)} - X^\delta_{t(N)} - X_s + X^\delta_s|(t-s)^{-1-\gamma} ds\right)^2.$$

Theorem 3. *Let assumptions (A)–(D) hold and equation* (1) *has the unique solution* X*. Then the Euler approximations* X^δ *converge to* X *in Besov space* $W^3_\gamma[0,T]$ *for any* $0 < \gamma < 1/2$ *in the following sense: for any* $\eta > 0$ *there exists* $C = C_\eta$ *such that*

$$||X^\delta - X||_{\gamma,N} \le \exp\{CN^2\}\delta^{\kappa+H-1-\eta},$$

where $\kappa = \min\{1/2, \beta\}$*.*

Mixed financial markets

2. We consider financial market with risky asset governed by both the Wiener process and fractional Brownian motion with Hurst parameter $H > 3/4$. Using Hitsuda and Cheridito representations for the mixed Brownian–fractional Brownian process, we present the solution of the problem of efficient hedging.

Let we have a financial market with two assets: non-risky asset

$$B_t = B_0 e^{rt}, \quad t \geq 0, \quad B_0 > 0, \tag{6}$$

$r > 0$ is a constant risk-free rate, and risky asset that is governed by the linear combination of W and B^H

$$S_t = S_0 \exp\{\mu t + \sigma_1 W_t + \sigma_2 B_t^H\}, \quad t \geq 0, \tag{7}$$

where $S_0 > 0$, $\mu \in \mathbb{R}$ is a drift coefficient, $\sigma_1 > 0$ is a volatility coefficient for standard Brownian motion W, $\sigma_2 > 0$ is a volatility coefficient for fBm B^H. Such model will be called the mixed Brownian-fractional-Brownian one.

Fix some finite horizon $T > 0$ and consider our market on the interval $[0, T]$. Denote the filtration $\mathbb{F}^S = \{\mathbb{F}^S_t, 0 \le t \le T\}$, where $\mathbb{F}^S_t = \sigma\{S_u, 0 \le u \le t\}$. Further, by $\bar{\mathbb{F}}^S = \{\bar{\mathbb{F}}^S_t, 0 \le t \le T\}$ we denote the smallest filtration that contains $\mathbb{F}^S$ and fulfils the usual assumptions. The following properties of the model (6) and (7) were established by Hitsuda and Cheridito:

1. The mixed process $M_t^{H,\sigma} = W_t + \sigma B_t^H$, $t \in [0, T]$ is equivalent (in measure) to Brownian motion if and only if $H \in (3/4, 1]$.
2. For $H \in (3/4, 1]$ there exists a unique real-valued Volterra kernel $\widetilde{r}_\sigma \in \mathbb{L}_2([0, T]^2)$ such that

$$B_t := M_t^{H,\sigma} - \int_0^t \int_0^s \widetilde{r}_\sigma(s, u) dM_u^{H,\sigma} ds, \quad t \in [0, T] \tag{8}$$

is a Brownian motion on $(\Omega, \mathbb{F}, P)$.

3. The "inverse" representation holds:

$$M_t^{H,\sigma} := B_t + \int_0^t \int_0^s r_\sigma(s,u) dB_u ds, \quad t \in [0,T],$$

where $r_\sigma \in \mathbb{L}_2([0,T]^2)$ is the negative resolvent kernel of $\widetilde{r}_\sigma$, r_σ is the unique solution of the equation

$$\sigma^2 H(2H-1)(t-s)^{2H-2} = r_\sigma(t,s) + \int_0^s r_\sigma(t,x) r_\sigma(s,x) dx, \quad 0 \le s < t \le T, \tag{9}$$

and this representation is unique in the following sense: if $\widetilde{B}_t$ is a Brownian motion on $(\Omega, \mathbb{F}, P)$ and $l \in \mathbb{L}_2([0,T]^2)$ a real-valued Volterra kernel such that

$$M_t^{H,\sigma} := \widetilde{B}_t + \int_0^t \int_0^s l(s,u) d\widetilde{B}_u ds, \quad t \in [0,T],$$

then $\widetilde{B} = B$ and $l = r_\sigma$.

4. As a consequence, the process $\sigma_1 W_t + \sigma_2 B_t^H$ is a semimartingale with respect to its natural filtration. Let the process $\Psi(s)$ be $\bar{\mathbb{F}}^S$-predictable and satisfy the condition $\int_0^T |\Psi_u|^2 du < \infty$ a.s. Then the stochastic integral $\int_0^T \Psi_u dS_u$ is correctly defined as the integral w.r.t. the semimartingale.

5. Let $\{B_t, t \in [0,T]\}$ be a Brownian motion on a probability space $(\Omega, \mathbb{F}, P)$ and $l \in \mathbb{L}_2([0,T]^2)$ a real-valued Volterra kernel. Then

$$E \exp\left(\int_0^t \int_0^s l(s,u)dB_u dB_s - \frac{1}{2}\int_0^t \left(\int_0^s l(s,u)dB_u\right)^2 ds\right) = 1$$

moreover, by Girsanov theorem,

$$B_t - \int_0^t \int_0^s l(s,u)dB_u ds, \quad t \in [0,T]$$

is a Brownian motion on $(\Omega, \mathbb{F}, \widetilde{P})$, where

$$\frac{d\widetilde{P}}{dP} = \exp\left(\int_0^T \int_0^s l(s,u)dB_u dB_s - \frac{1}{2}\int_0^T \left(\int_0^s l(s,u)dB_u\right)^2 ds\right).$$

6. If we consider the following class of strategies

$$\mathbb{S} = \{\Psi = (\Psi^1, \Psi^2)\} : \quad \Psi^1 \quad \text{and}$$

$$\Psi^2 \quad \text{are} \quad \mathbb{F}^{\mathbb{S}} \quad \text{-predictable,}$$

$$\int_0^T |\Psi_u^1| du < \infty, \quad \int_0^T |\Psi_u^2|^2 du < \infty \quad \text{a.s.,}$$

$$V_t = V_0 + \int_0^t \Psi_u^2 dS_u, \quad t \in [0, T]$$

and there exists a constant $c \geq 0$ such that

$$\inf_{t \in [0,T]} \int_0^T \Psi_u^2 dS_u \geq -c \quad \text{a.s}\},$$

then the model (6)–(7) is arbitrage-free and complete.

Now we establish one auxiliary result concerning the form of the kernel $r_\sigma(t,x)$.
Lemma 1. *For any* $0 \le s \le t \le T \wedge T\sigma^{-1/\alpha}$ *we have that*

$$r_\sigma(t,s) = \sigma^{1/\alpha} r_1(\sigma^{1/\alpha}t, \sigma^{1/\alpha}s),$$

$\alpha = H - 1/2.$

Now, denote $\sigma = \frac{\sigma_2}{\sigma_1}$.

Lemma 2. *There exists such constants $C > 0$ and $\varepsilon > 0$ that for any $\sigma < C$ we have the relation*

$$E\exp\left\{\frac{1+\varepsilon}{2}\int_0^T\left(\int_0^s r_\sigma(s,u)dB_u\right)^2 ds\right\} < \infty \tag{10}$$

whence for any $\sigma < C$ there exists the unique probability measure $P^* = P^*(\sigma)$ such that the discounted process

$$S_t\exp\{-rt\} = S_0\exp\left\{(\mu-r)t + \sigma_1\left(W_t + \sigma B_t^H\right)\right\} =$$

$$S_0\exp\left\{(\mu-r)t + \sigma_1\left(B_t + \int_0^t\int_0^s r_\sigma(s,u)dB_u ds\right)\right\}$$

becomes a martingale with respect to P^* and to the natural filtration $\bar{\mathbb{F}}^S$.

If we intend to choose such probability measure P^* that the discounted process

$$S_t \exp\{-rt\} = S_0 \exp\left\{(\mu - r)t + \sigma_1 \left(B_t + \int_0^t \int_0^s r_\sigma(s,u) dB_u ds\right)\right\}$$

becomes P^*-martingale, then $\frac{dP^*}{dP}$ is defined by the unique relation

$$\frac{dP^*}{dP}\Big|_{\bar{\mathbb{F}}_t^S} = \exp\Big\{ -\int_0^t \Big(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u) dB_u\Big) dB_s - \frac{1}{2}\int_0^t \Big(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u) dB_u\Big)^2 ds\Big\}. \tag{11}$$

It is established throughout the proof that the theorem is valid for

$$C_{H,T}\sigma^4 \exp\{2C_T\} < \frac{1}{1+\varepsilon}, \quad (12)$$

where $C_{H,T} = 2H^2(2H-1)^2\frac{T^{4H-3}}{4H-3}$ depends only on H and T, and

$$C_T = \int_0^T \int_0^s r_1^2(s,x)dxds.$$

The problem of efficient hedging on the mixed market

Let H be some contingent claim on our financial market. Consider the natural filtration $\bar{\mathbb{F}}^S$ generated by the process $W_t + \sigma B_t^H$, $0 \le t \le T$ and the class $\mathbb{S}$ of self-financing strategies described above. Consider the problem of efficient hedging, which purpose is to minimize the potential losses, weighted by the hedger's risk preference from imperfect hedging. Efficient hedging aims at finding an admissible self-financing strategy $\Psi^* = (\Psi^{*,1}, \Psi^{*,2}) \in \mathbb{S}$ that minimizes the shortfall risk

$$E\left(l\left(\left(H - V_T^{\Psi^*}\right)^+\right)\right) = \min_{\Psi} E\left(l\left(\left(H - V_T^{\Psi^*}\right)^+\right)\right)$$

with initial capital $V_0 \le \nu_0 < H_0 := E_{P^*}(He^{-rT})$. Here l denotes the loss function that reflects the investor's risk preference.

According to Foellmer and Leukert, the important particular case is the loss function $l(x) = \frac{x^p}{p}$, $p > 0$, where $p > 1$ corresponds to risk-averse investor, $p = 1$ corresponds to risk indifference, and $0 < p < 1$ means that the investor is risk-taker. In the general case, the minimization problem for some set of measures $P^* \in \mathcal{P}$ can be reformulated as follows (Foellmer, Leukert):

(A) to find the randomized test, or minimizer, $0 \le \psi^* \le 1$, which is $\bar{\mathbb{F}}_T$-measurable, and which minimizes the shortfall risk $E[l(H(1-\psi))]$ among all $\bar{\mathbb{F}}_T$-measurable $0 \le \psi \le 1$ subject to constraints $E^* \psi H \le \nu_0$ for all $P^* \in \mathcal{P}$.

Denote $I = (l')^{-1}$ the inverse function to l'. It was proved by Foellmer, Leukert that the solution of the problem of imperfect efficient hedging is the perfect hedge Ψ^* for the modified contingent claim $H^* = \varphi^* H$, where φ^* is determined as

$$\begin{aligned} \varphi^* &= 1 - \left(\frac{I\left(a^* e^{-rT} Z_T\right)}{H} \wedge 1 \right) \quad \text{for} \quad p > 1, \\ \varphi^* &= I\left\{ a^* e^{-rT} H^{1-p} Z_T \right\} \quad \text{for} \quad 0 < p < 1, \\ \varphi^* &= 1\left\{ a^* e^{-rT} Z_T < 1 \right\} \quad \text{for} \quad p = 1, \end{aligned} \tag{13}$$

and Z_T refers to the density of equivalent martingale measure P^*:

$$Z_t = \frac{dP^*}{dP}\bigg|_{\bar{\mathbb{F}}_t^S}, \quad t \in [0, T],$$

a^* is such constant that $E_{P^*}[H\varphi^*] = \nu_0$.

The formula (13) gives the general solution of our problem of imperfect efficient hedging for small σ, satisfying (12). However, it is hard to proceed with some computations because the distribution of Z_T, according to (11), depends on the whole trajectory of Wiener process $\{B_t, 0 \leq t \leq T\}$. However, in turn, in the case when the objective measure P coincides with the measure P^* and the contingent claim H depends only on the final value of discounted risk asset: $H = H(X_T)$, the situation can be simplified.

Indeed, in this case

$$E\left(l\left(\left(H - V_T^{\Psi}\right)^+\right)\right) = E\left(\left(H(X_T) - \psi_T e^{-rT} - \psi_T^2 X_T\right)^+\right)^p,$$

where $E(\cdot)$ means mathematical expectation with respect to the measure for which X_T has the known log-normal distribution, $X_T = S_0 \exp\left\{\sigma_1 B_T - \frac{\sigma_1^2}{2} T\right\}$, B is Wiener process. The condition $E_{P^*} H \le \nu_0$ is reduced to $EH(X_T) \le \nu_0$, and we come to the standard problem of efficient hedging.

Imperfect hedging on an incomplete market. Approximations of fractional Brownian motion. Minimal martingale measure for an approximate market

Incomplete semimartingale market,
constructed as an approximation of initial market

For some reasons, we can try to apply another approach to the solution of the problem of efficient hedging. One of possible reasons is: let the objective measure P does not coincide with the martingale measure P^* but we still want to obtain comparatively simple and computable distribution of the solution of the problem of efficient hedging.

Another reason: even if $H \in (\frac{1}{2}, \frac{3}{4})$, we can try to solve the problem of efficient hedging, using the properties of the involved processes W and B^H. In this case we can consider an incomplete market adapted to the filtration generated by the couple of two independent processes W and fBm B^H. Since B^H is not a semi-martingale we have no martingale measure P^*. However, there are some possibilities to approximate B^H with the help of the bounded processes of bounded variation. One of these possibilities was considered by Androshchuk.

Consider the representation of fBm B^H via some Wiener process $\tilde{W}$ on the finite interval $[0, T]$ (Norros, Valkeila, Virtamo):

$$B_t^H = C_H^1 \int_0^t s^{-\alpha} \left(\int_s^t u^{\alpha}(u-s)^{\alpha-1} du \right) d\tilde{W}_s, \qquad (14)$$

where $C_H^1 = \alpha \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)} \right)^{1/2}$, $\alpha = H - \frac{1}{2}$, Wiener processes $\tilde{W}$ and W are independent. If we formally apply stochastic Fubini theorem to the right-hand side of (14), we obtain interior integral divergent, due to singularity in the upper limit of integration.

However, if we retreat from the singularity point, we can obtain the family of bounded processes of bounded variation $B_t^{H,\varepsilon}$ of the form

$$B_t^{H,\varepsilon} = \int_0^t \varphi_\varepsilon(s)ds,$$

where

$$\varphi_\varepsilon(s) = (C_H^1 s^\alpha \int_0^{(1-\varepsilon)s} u^{-\alpha}(s-u)^{\alpha-1} d\tilde{W}_u) \wedge (\varepsilon)^{-1}, \quad 0 < \varepsilon < 1.$$

As it follows from the results of Androshchuk, $B_t^{H,\varepsilon} \xrightarrow{P} B_t^H$ as $\varepsilon \to 0$. Moreover, in the paper of Ralchenko and Shevchenko the convergence in probability of $B_t^{H,\varepsilon}$ to B_t in some Besov spaces, introduced above, was established. Now, consider the financial market with the same non-risky asset as in (6) and "approximate" risky asset

$$S_t = S_0 \exp\{\mu t + \sigma_1 W_t + \sigma_2 B_t^{H,\varepsilon}\}, \quad t \geq 0. \tag{15}$$

This approximate price of risky asset is a semi-martingale with respect to the filtration $\mathbb{F}_t^{W,\tilde{W}} := \sigma\{W_s, \tilde{W}_s, 0 \leq s \leq t\},\ t \in [0, T]$.

By $\bar{\mathbb{F}}^{W,\tilde{W}} = \{\bar{\mathbb{F}}_t^{W,\tilde{W}}, 0 \leq t \leq T\}$ we denote the smallest filtration that contains $\mathbb{F}^{W,\tilde{W}}$ and fulfils the usual assumptions. Introduce also the notation $\mathbb{F}_t^{\tilde{W}} := \sigma\{\tilde{W}_s, 0 \leq s \leq t\},\ t \in [0, T]$ and corresponding filtration $\bar{\mathbb{F}}^{\tilde{W}}$. The next result is an evident consequence of the structure of approximate market.

Lemma 3. *All equivalent martingale measures P^* for the market described by the equations* (6) *and* (15)*, equal a product of two likelihood ratios*

$$\frac{dP^*}{dP}\bigg|_{\bar{\mathbb{F}}_t^{W,\tilde{W}}} = Z_t^{\varepsilon,1} Z_t^2, \tag{16}$$

where

$$Z_t^{\varepsilon,1} = \exp\left\{\int_0^t \theta_\varepsilon(s) dW_s - \frac{1}{2}\int_0^t \theta_\varepsilon^2(s) ds\right\}, \tag{17}$$

$$\theta_\varepsilon(s) = \frac{r-\mu}{\sigma_1} - \frac{\sigma_1}{2} - \frac{\sigma_2}{\sigma_1}\varphi_\varepsilon(s), \tag{18}$$

$$Z_t^2 = \exp\left\{\int_0^t b(s) d\tilde{W}_s - \frac{1}{2}\int_0^t b^2(s) ds\right\}, \tag{19}$$

where $b = b(s)$ be any $\bar{\mathbb{F}}_s^{W,\tilde{W}}$-adapted function such that $EZ_T^{\varepsilon,1} Z_T^2 = 1$.

Minimal martingale measure (MMM) on the approximate market

Evidently, among all the measures, described by the relations (16) – (19), the simplest measure corresponds to the case $b(s) \equiv 0$, or $Z_t^2 \equiv 1$. Consider this measure in more detail.

Definition 1. (Foellmer, Schweizer) Probability measure $P^* \sim P$ is called minimal martingale measure if $E\left(\frac{dP^*}{dP}\right)^2 < \infty$ and any square-integrable P-martingale M, strongly orthogonal to the discounted price process $X_t = e^{-rt} S_t$, is also P^*-martingale.

According to the paper of Schweizer, minimal martingale measure can be found in the following way. Let the semi-martingale Y has a canonical decomposition of the form

$$Y = Y_0 + M + A = X_0 + M + \int \alpha d\langle M\rangle, \tag{20}$$

where M is a martingale, α is the predictable process, then the minimal martingale measure has a density

$$\widehat{Z}_T = \exp\left\{-\int_0^T \alpha_s dM_s - \frac{1}{2}\int_0^T \alpha_s^2 d\langle M\rangle_s\right\},$$

under the condition $E\widehat{Z}_T^2 < \infty$.

In our case, Y is discounted price process

$$X_t^\varepsilon = S_0 \exp\left\{(\mu - r)t + \sigma_1 W_t + \sigma_2 \int_0^t \varphi_\varepsilon(s)ds\right\},$$

and, according to Itô formula,

$$X_t^\varepsilon = S_0 + \sigma_1 \int_0^t X_s^\varepsilon dW_s + \frac{\sigma^2}{2}\int_0^t X_s^\varepsilon ds + \int_0^t X_s^\varepsilon[(\mu - r) + \sigma_2\varphi_\varepsilon(s)]ds.$$

Therefore, we can put in (20)

$$M_t = \sigma_1 \int_0^t X_s dW_s, \quad \langle M \rangle_t = \sigma_1^2 \int_0^t X_s^2 ds$$

and

$$\alpha_t = X_t^{-1} \left(\frac{1}{2} + \frac{\mu - r}{\sigma_1^2} + \frac{\sigma_2 \varphi_\varepsilon(t)}{\sigma_1} \right).$$

At last, it means that the density of MMM for approximate market has a form

$$\widehat{Z}_T = \widehat{Z}_T^\varepsilon = \exp \left\{ - \int_0^T \left(\frac{\sigma_1}{2} + \frac{\mu - r}{\sigma_1} + \sigma_2 \varphi_\varepsilon(s) \right) dW_s \right.$$
$$\left. - \frac{1}{2} \int_0^T \left(\frac{\sigma_1}{2} + \frac{\mu - r}{\sigma_1} + \sigma_2 \varphi_\varepsilon(s) \right)^2 ds \right\} = Z_T^{\varepsilon,1},$$

and minimal martingale measure (if it exists) corresponds to the case $b \equiv 0$ in (19), or $Z_t^2 \equiv 1$.

Only, we must check that $E(\widehat{Z}_T^{\varepsilon})^2 < \infty$. Prove the following auxiliary result.

Lemma 4. *The density $\widehat{Z}_T^{\varepsilon}$ has finite moments of any order.*

Proof.

Recall that $\theta_{\varepsilon}(s)$ is bounded and take into account independence of $\theta_{\varepsilon}(s)$ and W. Then we obtain that for any $q > 0$

$$E(\widehat{Z}_T^{\varepsilon})^q = E[E((\widehat{Z}_T^{\varepsilon})^q/\bar{\mathbb{F}}_T^{\tilde{W}})] = E\left(\frac{q^2-q}{2}\int_0^T \theta_{\varepsilon}^2(s)ds\right) < \infty. \qquad (21)$$

□

So, in our situation we choose minimal martingale measure as "the best" equivalent martingale measure.

The solution of the "approximate" problem of efficient hedging with respect to MMM

Let φ_ε^* be the function from (13), where we substitute $\widehat{Z}_T^\varepsilon$ instead of Z_T. In what follows we suppose that the discounted contingent claim H is positive and depends only on the final value of discounted price process, $H = H(X_T^\varepsilon)$, where $H(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is measurable function, and in this case present some results concerning simplifying of possible computations.
Lemma 5. *The equality holds,*

$$E\widehat{Z}_T^\varepsilon H(X_t^\varepsilon) = (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}} H(e^{\sigma_1 T^{\frac{1}{2}} x - \frac{\sigma_1^2}{2} T}) e^{\frac{-x^2}{2}} dx, \tag{22}$$

under the condition that the last integral is finite.

In what follows we consider only the case of risk-averse investor, when $p > 1$ in (13) (other cases can be considered along the same lines). So, throughout this section $l(x) = x^p$, $p > 1$. We suppose that $EH^p < \infty$. At first note that it follows from (21) that in this case both parts of equation (22) are finite.

Introduce the following notations. Let $\varsigma(\cdot) : \mathbb{R} \to \mathbb{R}$ be some measurable bounded function, $I_1(\varsigma) = \int_0^T \varsigma(s)ds$, $I_2(\varsigma) = \int_0^T \varsigma^2(s)ds$,
$H_1(x) = H(S_0 \exp\{\sigma_1 T^{\frac{1}{2}} x - \frac{\sigma_1^2}{2} T\})$, $G(\varsigma(\cdot), y) = \exp\{y + \frac{1}{2} \int_0^T \varsigma^2(s)ds\}$, and the matrix

$$P(\varsigma(\cdot)) = \begin{pmatrix} T & \int_0^T \varsigma(s)ds \\ \int_0^T \varsigma(s)ds, & \int_0^T \varsigma^2(s)ds \end{pmatrix}$$

is non-degenerate. Also, let the set

$$A(\varsigma(\cdot), a) = \{(x, y) \in \mathbb{R}^2 : a < H_1(x)^{p-1} G(\varsigma(\cdot), y)^{-1}\},$$

the function

$$\tilde{\Psi}(\varsigma(\cdot), a) = \int_{A(\varsigma(\cdot), a)} (H_1(x) - a^{\frac{1}{p-1}} G(\varsigma(\cdot), y)^{\frac{1}{p-1}}) p(x, y, \varsigma(\cdot)) dxdy,$$

$p(x, y, \varsigma(\cdot))$ is the density of bivariate Gaussian distribution with zero mean and covariance matrix $P(\varsigma(\cdot))$.

Lemma 6. *Suppose that the right-hand side of* (22) *is finite and let* a_ε^* *be the unique root of the equation*

$$E\tilde{\Psi}(\theta_\varepsilon(\cdot), a_\varepsilon^*) = \nu_0. \tag{23}$$

Then the solution (13) *of the minimization problem with* $p > 1$ *has a form*

$$\varphi_\varepsilon^* = 1 - \frac{(a_\varepsilon^*)^{\frac{1}{p-1}} (\widehat{Z}_T^\varepsilon)^{\frac{1}{p-1}}}{H(X_T^\varepsilon)} \wedge 1. \tag{24}$$

Asymptotic behavior of the solution of the problem of efficient hedging with respect to MMM

Now, let parameter value $\varepsilon \to 0$.
Lemma 7. *Under conditions of Lemma 6 we have that* $E[l(H(1-\varphi_\varepsilon^*))] \to 0$ *as* $\varepsilon \to 0$.
Remark 2. Relation $E[l(H(1-\varphi_\varepsilon^*))] \to 0$ means that under the filtration $\bar{\mathbb{F}}^{W,\tilde{W}}$ hedging of contingent claim H is asymptotically perfect for any $\nu_0 > 0$.

The estimate of the solution of the "approximate" problem of efficient hedging for mixed measures

Since is hard to deduce computable formula for the solution of the "approximate" problem of efficient hedging with respect to the whole family of the "mixed" measures defined by (16)–(19), we restrict ourself to this family with $\bar{\mathbb{F}}_s^{\tilde{W}}$-adapted process $b \in L_2([0,T] \times \Omega)$. Denote

$$K(\varsigma(\cdot)) = (H(X_T^\varepsilon) - (a_\varepsilon^*)^{\frac{1}{p-1}}(\widehat{Z}_T^\varepsilon(\varsigma(\cdot)))^{\frac{1}{p-1}})1_{\{(a_\varepsilon^*)^{\frac{1}{p-1}}(\widehat{Z}_T^\varepsilon(\varsigma(\cdot)))^{\frac{1}{p-1}} \le H(X_T^\varepsilon)\}}, \tag{25}$$

where we substitute $\varsigma(\cdot)$ instead of $\theta_\varepsilon(\cdot)$ into $\widehat{Z}_T^\varepsilon$, i.e.,
$\widehat{Z}_T^\varepsilon(\varsigma(\cdot)) = \exp\{-\int_0^T \varsigma(s)dW_s - \frac{1}{2}\int_0^T \varsigma(s)^2 ds\}$.
Consider the equation

$$E_{P^*} K(\varsigma(\cdot)) = \nu_0, \tag{26}$$

where the function $K(\varsigma(\cdot))$ is defined in (25). The equation (26) has the unique solution $a_\varepsilon^* = a_\varepsilon^*(\varsigma(\cdot))$.

Lemma 8. *Let* ψ_ε *be the minimizer for the problem of the efficient hedging for the family of measures defined by* (16)–(19) *with* $\bar{\mathbb{F}}_s^{\tilde{W}}$*-adapted processes* $b \in L_2([0,T] \times \Omega)$. *Then the upper bound holds,*

$$E[l(H(1-\psi_\varepsilon^*))] \le E[l(H(1-\varphi_\varepsilon^*(\theta_\varepsilon(\cdot))))].$$

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