Multivariate supOU processes and a stochastic volatility model with possible long memory

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Motivation and Idea

Stylized Facts of Financial Time Series

- non-constant, stochastic volatility (variance)
- volatility exhibits jumps
- asymmetric and heavily tailed marginal distributions
- clusters of extremes
- log returns exhibit marked dependence, but have vanishing autocorrelations (squared returns, for instance, have non-zero autocorrelation)
- **long memory**

Stochastic Volatility Models are used to cover these stylized facts.
But: **Long memory hard to obtain**

Our Aim: A multivariate stochastic volatility model with long memory and high analytic tractability

Stationary univariate Ornstein-Uhlenbeck processes

Let L be a univariate Lévy subordinator with $E(\ln^+(L_1)) < \infty$ and $a < 0$. Then the integrals

$$\sigma_t = \int_{-\infty}^{t} e^{a(t-s)} dL_s$$

are well-defined (ω-wise) and the process σ is a stationary positive Ornstein-Uhlenbeck type process.

Provided $\mathbb{V}\mathrm{ar}(L_1) < \infty$, we have

$$\mathbb{C}\mathrm{ov}(\sigma_h, \sigma_0) = e^{ah}\mathbb{V}\mathrm{ar}(\sigma_0),$$

hence a short memory process.

Note: If L is a Lévy process of infinite variation $\int_{-\infty}^{t} e^{a(t-s)} dL_s$ exists as a limit in probability and gives again a stationary process.

Finite superposition of OU type processes

- Let σ_1 and σ_2 be two independent stationary positive OU type processes of finite variance with mean reversion coefficients a_1 and a_2.
- $\pi_1, \pi_2 \geq 0$ and $\pi_1 + \pi_2 = 1$
- The process $\sigma = \pi_1\sigma_1 + \pi_2\sigma_2$ is called a superposition of two OU type processes (supOU process).
- $\mathbb{C}\text{ov}(\sigma_h, \sigma_0) = \pi_1^2 e^{a_1 h} \mathbb{V}\text{ar}(\sigma_{1,0}) + \pi_2^2 e^{a_2 h} \mathbb{V}\text{ar}(\sigma_{2,0})$
- Assume w.l.o.g. $a_2 > a_1$.
  - For $h \to \infty$ we have $\mathbb{C}\text{ov}(\sigma_h, \sigma_0) \sim \pi_2^2 e^{a_2 h} \mathbb{V}\text{ar}(\sigma_{2,0})$.
  - Hence: Still a short memory process, asymptotic decay governed by slowest exponential decay rate
  - Initial decay of the autocovariance (i.e. close to zero) usually governed by faster exponential decay rate.
- The same holds for superpositions of n independent OU processes.

Infinite superposition of OU type processes

- Idea: Adding up infinitely many OU type processes with "eventually arbitrarily slow exponential decay" (i.e. a close to 0) may result in autocovariance with non-exponential decay.
- Extension: "Sum up" independent OU type processes with all possible mean-reversion speeds $a \in \mathbb{R}^-$ weighted by a probability measure π:

$$\text{“}\sigma_t = \int_{\mathbb{R}^-} \int_{-\infty}^{t} e^{a(t-s)} dL_s^{(a)} \pi(da)\text{”}; \quad \mathbb{C}\mathrm{ov}(X_h, X_0) = \int_{\mathbb{R}^-} \frac{e^{ah}\mathbb{V}\mathrm{ar}(L_1)}{2a} \text{“}\pi^2\text{”}(da)$$

  Example:
  "π^2" $= -C\Gamma(\alpha,\beta)$, $\alpha > 1$, $C > 0$:
  $\mathbb{C}\mathrm{ov}(X_h, X_0) = \frac{C\beta^\alpha}{2(\alpha-1)}(\beta + h)^{1-\alpha}\mathbb{V}\mathrm{ar}(L_1)$
- Need to address this in a rigorous mathematical manner.
- Intuitive idea: Different "News" are forgotten at different exponential rates. Some news are forgotten very slowly $\Rightarrow$ Long-range dependence
- Alternative to long memory via "fractional integration".

Some matrix notation

- $M_d(\mathbb{R})$: the real $d \times d$ matrices.
- $\mathbb{S}_d$: the real symmetric $d \times d$ matrices.
- $\mathbb{S}_d^+$: the positive semi-definite $d \times d$ matrices (covariance matrices) (a closed cone).
- $\mathbb{S}_d^{++}$: the positive definite $d \times d$ matrices (an open cone).
- $A^{1/2}$: for $A \in \mathbb{S}_d^+$ the unique positive semi-definite square root (functional calculus).
- $\mathrm{tr}(A)$: The trace of a matrix A.

Positive semi-definite OU type processes

Positive semi-definite OU type processes

Theorem

Let $(L_t)_{t\in\mathbb{R}}$ *be a matrix subordinator with* $E(\max(\log\|L_1\|,0))<\infty$ *and* $A\in M_d(\mathbb{R})$ *such that* $\sigma(A)\subset(-\infty,0)+i\mathbb{R}$.
Then the stochastic differential equation of Ornstein-Uhlenbeck-type

$$d\Sigma_t=(A\Sigma_{t-}+\Sigma_{t-}A^T)dt+dL_t$$

has a unique stationary solution

$$\Sigma_t=\int_{-\infty}^t e^{A(t-s)}dL_s e^{A^T(t-s)}$$

or, in vector representation,
$\mathrm{vec}(\Sigma_t)=\int_{-\infty}^t e^{(I_d\otimes A+A\otimes I_d)(t-s)}d\mathrm{vec}(L_s)$.
Moreover, $\Sigma_t\in\mathbb{S}_d^+$ *for all* $t\in\mathbb{R}$.

Positive semi-definite supOU processes

Lévy basis (i.d.i.s.r.m.)

$M_d^- := \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R}\}$
$\mathcal{B}_b\left(M_d^- \times \mathbb{R}\right)$: the bounded Borel sets of $M_d^- \times \mathbb{R}$.

Definition

A family $\Lambda = \{\Lambda(E) : E \in \mathcal{B}_b\left(M_d^- \times \mathbb{R}\right)\}$ of $\mathbb{S}_d^+$-valued random variables is called a positive semi-definite Lévy basis on $M_d^- \times \mathbb{R}$ if:

1. the distribution of $\Lambda(E)$ is infinitely divisible for all $E \in \mathcal{B}_b\left(M_d^- \times \mathbb{R}\right)$,
2. for any $n \in \mathbb{N}$ and pairwise disjoint sets $E_1, \dots, E_n \in \mathcal{B}_b\left(M_d^- \times \mathbb{R}\right)$ the random variables $\Lambda(E_1), \dots, \Lambda(E_n)$ are independent and
3. for any pairwise disjoint sets $E_i \in \mathcal{B}_b\left(M_d^- \times \mathbb{R}\right)$ with $i \in \mathbb{N}$ satisfying $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{B}_b\left(M_d^- \times \mathbb{R}\right)$ the series $\sum_{n=1}^{\infty} \Lambda(E_n)$ converges almost surely and $\Lambda\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} \Lambda(E_n)$.

Lévy basis (i.d.i.s.r.m.)

We consider only Lévy bases having characteristic function of the form

$$E\left(\exp(i\operatorname{tr}(u\Lambda(E)))\right) = \exp\left(\psi(u)\Pi(E)\right)$$

for all $u \in \mathbb{S}_d$ and $E \in \mathcal{B}_b\left(M_d^-(\mathbb{R}) \times \mathbb{R}\right)$,
where $\Pi = \pi \times \lambda$ is the product of a probability measure π on $M_d^-(\mathbb{R})$ and the Lebesgue measure λ on $\mathbb{R}$.
Moreover,

$$\psi(u) = i\operatorname{tr}(u\gamma) + \int_{\mathbb{S}_d^+} \left(e^{i\operatorname{tr}(ux)} - 1\right) \nu(dx)$$

is the cumulant transform of an infinitely divisible distribution on $\mathbb{S}_d^+$ with Lévy-Khintchine triplet $(\gamma, 0, \nu)$.

$$L_t = \Lambda(M_d^- \times (0,t]) \text{ and } L_{-t} = \Lambda(M_d^- \times (-t,0)) \text{ for } t \in \mathbb{R}^+$$

is a Lévy process with characteristic triplet $(\gamma, 0, \nu)$ and it is called "the underlying Lévy process".

Positive semi-definite supOU processes

Theorem

Assume:

- $\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty$
- *There exist measurable functions* $\rho : M_d^- \to \mathbb{R}^- \setminus \{0\}$ *and* $\kappa : M_d^- \to [1, \infty)$ *such that:*

$$\|e^{As}\| \le \kappa(A) e^{\rho(A)s} \; \forall\, s \in \mathbb{R}^+,\ \pi\text{−a.s.}\ ,\ -\int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty.$$

Then the process $(\Sigma_t)_{t\in\mathbb{R}}$ *given by*

$$\Sigma_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds) e^{A^T(t-s)}$$

is well-defined for all $t \in \mathbb{R}$ *and* $\omega \in \Omega$ *and* Σ *is stationary.*
$\Sigma_t \in \mathbb{S}_d^+$ *for all* $t \in \mathbb{R}$*.*

Stationary distribution

The distribution of Σ_t is infinitely divisible with characteristic function

$$E\left(\exp\left(i\mathrm{tr}(u\Sigma_t)\right)\right) = \exp\left(i\mathrm{tr}(u\gamma_{\Sigma,0}) + \int_{\mathbb{S}_d}\left(e^{i\mathrm{tr}(ux)} - 1\right)\nu_\Sigma(dx)\right),\ u \in \mathbb{S}_d,$$

where

$$\gamma_{\Sigma,0} = \int_{M_d^-}\int_0^\infty e^{As}\gamma_0 e^{A^Ts} ds\pi(dA),$$

$$\nu_\Sigma(E) = \int_{M_d^-}\int_0^\infty\int_{\mathbb{S}_d^+} 1_E(e^{As}xe^{A^Ts})\nu(dx)ds\pi(dA)\ \forall\, E \subseteq \mathcal{B}(\mathbb{S}_d).$$

Restricting A to normal matrices

The condition

$$\|e^{As}\| \leq \kappa(A) e^{\rho(A)s} \; \forall \, s \in \mathbb{R}^+, \, \pi - \text{a.s.} \, , \; - \int_{M_d^-} \frac{\kappa(A)^2}{\rho(A)} \pi(dA) < \infty.$$

becomes:

- $-\int_{\mathbb{R}^-} \frac{1}{A} \pi(dA) < \infty$ in dimension 1 – the well-known necessary and sufficient existence condition for one-dimensional supOU processes. (cf. Barndorff-Nielsen (2001), Fasen and Klüppelberg (2007))
- For π concentrated on the normal (especially symmetric) matrices:

$$- \int_{M_d^-} \frac{1}{\max(\Re(\sigma(A)))} \pi(dA) < \infty$$

Necessary Conditions for the Existence of supOU Processes

$j(Z) = \min_{\|x\|=1} \|Zx\|$, $Z \in M_d(\mathbb{R})$, denotes the modulus of injectivity.

Proposition

Assume there exist measurable functions $\tau : M_d^- \to \mathbb{R}^+\backslash\{0\}$ *and* $\vartheta : M_d^- \to (0,1]$ *such that:* $j(e^{As}) \geq \vartheta(A)e^{-\tau(A)s}\ \forall\, s \in \mathbb{R}^+, \pi - a.s.$. *Then necessary conditions for the supOU integral to exist are:*

$$\int_{\vartheta(A)\geq\epsilon} \frac{1}{\tau(A)}\pi(dA) < \infty,\ \textit{for any}\ \epsilon \in (0,1]$$

$$\textit{such that}\ \nu(\{\|x\| > 1/\epsilon\}) > 0, \pi(\{\vartheta(A) \geq \epsilon\}) > 0,$$

$$\int_{M_d^-} \frac{\vartheta(A)^2}{\tau(A)}\pi(dA) < \infty,\ \textit{provided}\ j(\Sigma) > 0\ \textit{or}\ \nu(\{\|x\| \leq 1\}) > 0,\ \textit{and}$$

$$\int_{\|x\|>1} \ln(\|x\|)\nu(dx) < \infty.$$

“SDE representation” and path properties

Theorem

Provided

$$-\int_{M_d^-} \frac{(\|A\| \vee 1)\kappa(A)^2}{\rho(A)} \pi(dA) < \infty \text{ and } \int_{M_d^-} \|A\|\kappa(A)^2 \pi(dA) < \infty$$

it holds that

$$\Sigma_t = \Sigma_0 + \int_0^t Z_u du + L_t$$

where $L_t = \Lambda(M_d^- \times (0, t])$ *is a matrix subordinator and*

$$Z_u = \int_{M_d^-} \int_{-\infty}^{u} \left(A e^{A(u-s)} \Lambda(dA, ds) e^{A^T(u-s)} + e^{A(u-s)} \Lambda(dA, ds) e^{A^T(u-s)} A^T \right)$$

for all $u \in \mathbb{R}$ *with the integral existing* ω*-wise.*

$\Rightarrow$ *The paths of* Σ *are càdlàg and of finite variation on compacts.*

Second order moment structure

If $\int_{\|x\|>1} \|x\|^2 \nu(dx) < \infty, \quad -\int_{M_d^-} \frac{\kappa(A)^4}{\rho(A)} \pi(dA) < \infty$, then
$E(\|\Sigma_t\|^2) < \infty$ and

$$E(\Sigma_0) = -\int_{M_d^-} \mathbf{A}(A)^{-1} \left(\gamma + \int_{\mathbb{S}_d} x \nu(dx) \right) \pi(dA)$$

$$\mathbb{V}\mathrm{ar}(\mathrm{vec}(\Sigma_0)) = -\int_{M_d^-} (\mathcal{A}(A))^{-1} \left(\int_{\mathbb{S}_d} \mathrm{vec}(x) \mathrm{vec}(x)^T \nu(dx) \right) \pi(dA)$$

$$\mathbb{C}\mathrm{ov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0))$$
$$= -\int_{M_d^-} e^{(A \otimes I_d + I_d \otimes A)h} (\mathcal{A}(A))^{-1} \left(\int_{\mathbb{S}_d} \mathrm{vec}(x) \mathrm{vec}(x)^T \nu(dx) \right) \pi(dA), \ h \in \mathbb{R},$$

with $\mathbf{A}(A) : M_d(\mathbb{R}) \to M_d(\mathbb{R}), \ X \mapsto AX + XA^T$ and $\mathcal{A}(A)$:
$M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}), X \mapsto (A \otimes I_d + I_d \otimes A)X + X(A^T \otimes I_d + I_d \otimes A^T)$.
$\lim_{h\to\infty} \mathbb{C}\mathrm{ov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0)) = 0$.

An example with long memory

π: the distribution of RB with a diagonalisable $B \in M_d^-$ and R a real $\Gamma(\alpha,\beta)$-distributed random variable with $\alpha > 1, \beta \in \mathbb{R}^+ \backslash \{0\}$.
For the autocovariance function for positive lags h one obtains

$$\mathbb{C}\mathrm{ov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0))$$
$$= -\frac{\beta^\alpha}{\alpha - 1}\left(\beta I_{d^2} - (B \otimes I_d + I_d \otimes B)h\right)^{1-\alpha} \mathcal{B}^{-1}\left(\int_{\mathbb{S}^d} \mathrm{vec}(x)\mathrm{vec}(x)^T \nu(dx)\right)$$

with
$\mathcal{B} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}), X \mapsto (B \otimes I_d + I_d \otimes B)X + X(B^T \otimes I_d + I_d \otimes B^T)$.

- $\Rightarrow$ power decay in the autocovariance function
- For $\alpha \in (1,2)$: long memory.

A Stochastic Volatility Model

Let Σ be a positive semi-definite supOU process with càdlàg paths.

Then

$$Y_t = Y_0 + \int_0^t (\mu + \beta\Sigma_s)ds + \int_0^t \Sigma_s^{1/2} dW_s + \rho dL_t$$

with $\mu \in \mathbb{R}^d$, $\beta, \rho : \mathbb{S}_d \to \mathbb{R}^d$ linear and

$$L_t = \int_{M_d^-} \int_0^t \Lambda(dA, ds)$$

the underlying Lévy process is a well-defined d-dimensional stochastic volatility model.

The Autocovariance Structure of the Log-Returns and the Integrated Volatility

Assume $\mu, \beta, \rho = 0$, take $\Delta > 0$ and set

$$\mathbf{Y}_n = Y_{n\Delta} - Y_{(n-1)\Delta} \tag{1}$$

$$\mathbf{V}_n = \int_{(n-1)\Delta}^{n\Delta} \Sigma_s ds. \tag{2}$$

Then

$$\mathbb{C}\mathrm{ov}(\mathbf{Y}_1\mathbf{Y}_1^T, \mathbf{Y}_{h+1}\mathbf{Y}_{h+1}^T) = \mathbb{C}\mathrm{ov}(\mathbf{V}_1, \mathbf{V}_{h+1})$$

for all $h \in \mathbb{N}$.

The Integrated Volatility

Proposition

If

$$\int_{M_d^-} \kappa(A)^2 \pi(dA) < \infty,$$

the paths of Σ *are locally uniformly bounded in* t *for every* $\omega \in \Omega$. *Furthermore,* $\Sigma_t^+ = \int_0^t \Sigma_s ds$ *exists for all* $t \in \mathbb{R}^+$ *and*

$$\Sigma_t^+ = \int_{M_d^-} \int_{-\infty}^{t} (\mathbf{A}(A))^{-1} \left(e^{A(t-s)} \Lambda(dA, ds) e^{A^T(t-s)} \right)$$

$$- \int_{M_d^-} \int_{-\infty}^{0} (\mathbf{A}(A))^{-1} \left(e^{-As} \Lambda(dA, ds) e^{-A^T s} \right)$$

$$- \int_{M_d^-} \int_{0}^{t} (\mathbf{A}(A))^{-1} \Lambda(dA, ds)$$

with $\mathbf{A}(A) : \mathbb{S}_d \to \mathbb{S}_d, X \mapsto AX + XA^T$

Second Order Structure of V and YY^T I

Theorem

Let $\mu = \beta = \rho = 0$ and assume $\Sigma \in L^2$. Then $(\mathbf{V}_n)_{n\in\mathbb{N}}$ is stationary and square-integrable with

$$E(\mathbf{V}_1) = -\Delta \int_{M_d^-} \mathbf{A}(A)^{-1}\left(\gamma_0 + \int_{\mathbb{S}_d} x\nu(dx)\right)\pi(dA),$$

$$\mathbb{V}\mathrm{ar}(\mathrm{vec}(\mathbf{V}_1)) = r^{++}(\Delta) + r^{++}(\Delta)^*,$$

$$\mathbb{C}\mathrm{ov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) = r^{++}(h\Delta + \Delta) - 2r^{++}(h\Delta) + r^{++}(h\Delta - \Delta)$$

$$= -\int_{M_d^-} g(A,h)(\mathcal{A}(A))^{-1}\left(\int_{\mathbb{S}_d} \mathrm{vec}(x)\mathrm{vec}(x)^*\nu(dx)\right)\pi(dA),\ h \in \mathbb{N},$$

with $r^{++}(t) = \int_0^t \int_0^u \mathbb{C}\mathrm{ov}(\mathrm{vec}(\Sigma_s), \mathrm{vec}(\Sigma_0))dsdu$ *and*

$$g(A,h) = (A \otimes I_d + I_d \otimes A)^{-2} \cdot \left(e^{(A\otimes I_d + I_d \otimes A)(h\Delta+\Delta)} - 2e^{(A\otimes I_d + I_d\otimes A)h\Delta} + e^{(A\otimes I_d + I_d\otimes A)(h\Delta-\Delta)}\right).$$

It holds that $\lim_{h\to\infty} \mathbb{C}\mathrm{ov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) = 0.$

Second Order Structure of V and YY^T II

Theorem (Continued)

Likewise the log-price increments $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ as well as their "squares" $(\mathbf{Y}_n\mathbf{Y}_n^T)_{n\in\mathbb{N}}$ are stationary and square-integrable with

$$\begin{aligned}
E(\mathbf{Y}_1) &= 0, \quad \mathbb{V}\mathrm{ar}(\mathbf{Y}_1) = E(\mathbf{V}_1), \\
\mathbb{C}\mathrm{ov}(\mathbf{Y}_{h+1}, \mathbf{Y}_1) &= 0 \;\forall\; h \in \mathbb{N}, \\
E(\mathbf{Y}_1\mathbf{Y}_1^T) &= E(\mathbf{V}_1), \\
\mathbb{V}\mathrm{ar}(\mathrm{vec}(\mathbf{Y}_1\mathbf{Y}_1^T)) &= (I_{d^2} + \mathbf{Q} + \mathbf{PQ})\left(r^{++}(\Delta) + r^{++}(\Delta)^T\right) \\
&\quad + (I_{d^2} + \mathbf{P})\,(E(\mathbf{V}_1) \otimes E(\mathbf{V}_1))
\end{aligned}$$

$$\mathbb{C}\mathrm{ov}\left(\mathrm{vec}(\mathbf{Y}_{h+1}\mathbf{Y}_{h+1}^T),\; \mathrm{vec}(\mathbf{Y}_1\mathbf{Y}_1^T)\right) = \mathbb{C}\mathrm{ov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1)) \;\; \textit{for} \;\; h \in \mathbb{N}$$

where $\mathbf{P}, \mathbf{Q}$ *are certain linear operators.*

Long Memory in the SV Model?

Theorem

(i) If $\mathbb{C}\mathrm{ov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0))_{ij} \sim Ch^{-\alpha}$ *for* $h \to \infty$ *with* $\alpha > 0$ *and* $C \in \mathbb{R}\backslash\{0\}$, *then*

$$\mathbb{C}\mathrm{ov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1))_{ij} \sim C\Delta^{2-\alpha} h^{-\alpha} \text{ for } h \to \infty. \tag{3}$$

(ii) If $\mathbb{C}\mathrm{ov}(\mathrm{vec}(\Sigma_h), \mathrm{vec}(\Sigma_0))_{ij} \sim Ce^{-\alpha h}$ *with* $\alpha > 0$ *and* $C \in \mathbb{R}\backslash\{0\}$, *then*

$$\liminf_{h\to\infty} \left| \frac{\mathbb{C}\mathrm{ov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1))_{ij}}{C\Delta^2 e^{-\alpha(h\Delta+\Delta)}} \right| \geq 1, \tag{4}$$

$$\limsup_{h\to\infty} \left| \frac{\mathbb{C}\mathrm{ov}(\mathrm{vec}(\mathbf{V}_{h+1}), \mathrm{vec}(\mathbf{V}_1))_{ij}}{C\Delta^2 e^{-\alpha(h\Delta-\Delta)}} \right| \leq 1. \tag{5}$$

An example with long memory revisited

π: the distribution of RB with a diagonalisable $B \in M_d^-$ and R a real $\Gamma(\alpha,\beta)$-distributed random variable with $\alpha > 1, \beta \in \mathbb{R}^+ \backslash \{0\}$.

Then the squared returns have a polynomially decaying autocovariance

$$\mathbb{C}\mathrm{ov}(\mathbf{Y}_1\mathbf{Y}_1^T, \mathbf{Y}_{h+1}\mathbf{Y}_{h+1}^T)_{ij} \sim C_{ij} h^{1-\alpha}$$

and long memory if $\alpha \in (1,2)$.

This is far from obvious from the explicit formulae (with $\mathfrak{B} = (B \otimes I_d + I_d \otimes B)$):

$$\Gamma_h = \frac{\mathfrak{B}^{-2}\left((\beta I_{d^2} - \mathfrak{B}(h\Delta + \Delta))^{3-\alpha} - 2(\beta I_{d^2} - \mathfrak{B}h\Delta)^{3-\alpha} + (\beta I_{d^2} - \mathfrak{B}(h\Delta - \Delta))^{3-\alpha}\right)}{(2-\alpha)(3-\alpha)}, \quad \alpha \neq 2, 3$$

$$\Gamma_h = \mathfrak{B}^{-2}\Big((\beta I_{d^2} - \mathfrak{B}(h\Delta + \Delta))\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta + \Delta)) - 2(\beta I_{d^2} - \mathfrak{B}h\Delta)\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}h\Delta) + (\beta I_{d^2} - \mathfrak{B}(h\Delta - \Delta))\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta - \Delta))\Big), \quad \alpha = 2$$

$$\Gamma_h = \frac{\mathfrak{B}^{-2}\left(\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta + \Delta)) - 2\mathrm{Log}(\beta I_{d^2} - \mathfrak{B}h\Delta) + \mathrm{Log}(\beta I_{d^2} - \mathfrak{B}(h\Delta - \Delta))\right)}{(2-\alpha)}, \quad \alpha = 3.$$

Pricing in supOU models?

Pricing via Fourier techniques may be applicable, because

$$
\begin{aligned}
&E(e^{iY_t^T u}|\mathcal{G}_0) = \exp\Big\{i\Big[(Y_0+\mu t)^T u \\
&+\mathrm{tr}\Big(\int_{M_d^-}\int_{-\infty}^0 \mathbf{A}(A)^{-1}\Big(e^{A(t-s)}\Lambda(dA,ds)e^{A^T(t-s)} - e^{-As}\Lambda(dA,ds)e^{-A^T s}\Big)\Big(\beta^* u + \frac{i}{2}uu^T\Big)\Big)\Big] \\
&+\int_{M_d^-}\int_0^t \psi_\Lambda\Big[e^{A^T(t-s)}\Big(\mathbf{A}(A)^{-*}\Big(\beta^* u+\frac{i}{2}uu^T\Big)\Big)e^{A(t-s)} - \Big(\mathbf{A}(A)^{-*}\Big(\beta^* u+\frac{i}{2}uu^T\Big)-\rho^* u\Big)\Big]ds\pi(dA)\Big\}
\end{aligned}
$$

Does it really work and is it feasible? Future Research ....

Thank you very much for your attention!

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