Universal notions of independences

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- We do not assume that the product is commutative, i.e. generally $a \cdot b \neq b \cdot a$.
- We say that $\mathcal{A}$ is unital, if there exists a (necessarily unique) neutral element $\mathbf{1}$ with respect to the product.

The category of algebras with linear functionals

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In addition there are associated canonical embeddings:

$$\iota_A\colon \mathcal{A} \to \mathcal{A} \bullet \mathcal{B}, \quad \text{and} \quad \iota_B\colon \mathcal{B} \to \mathcal{A} \bullet \mathcal{B}.$$

Independence associated to a product "$\bullet$"

Consider a product "$\bullet$" on $\mathfrak{AwF}$.

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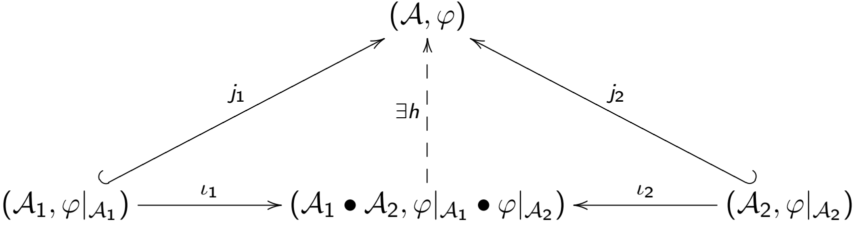

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Suppose $\mathtt{X}_1$ and $\mathtt{X}_2$ are two (real-valued) random variables on the classical probability space $(\Omega, \mathcal{F}, P)$.

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Consider also the two subalgebras given by

$$\mathcal{A}_1 = \{f(\mathtt{X}_1) \mid f \in \mathcal{B}_b(\mathbb{R})\}$$

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Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\otimes$-independent, i.e. there exists a mapping $h\colon \mathcal{A}_1 \otimes \mathcal{A}_2 \to (\mathcal{L}^\infty(\Omega, \mathcal{F}, P), \mathbb{E})$ such that the following diagram commutes:

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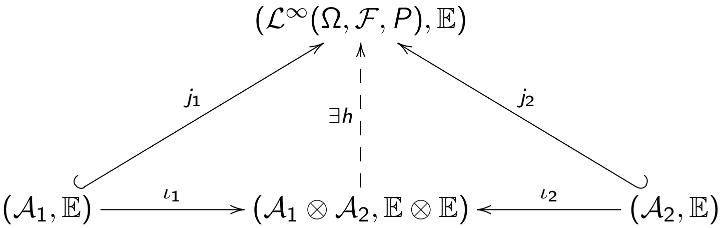

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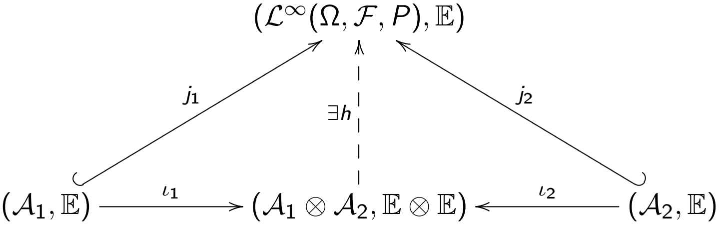

Here,

$$\iota_1(f(\mathtt{X}_1)) = f(\mathtt{X}_1) \otimes \mathbf{1}, \quad \text{and} \quad \iota_2(g(\mathtt{X}_2)) = \mathbf{1} \otimes g(\mathtt{X}_2),$$

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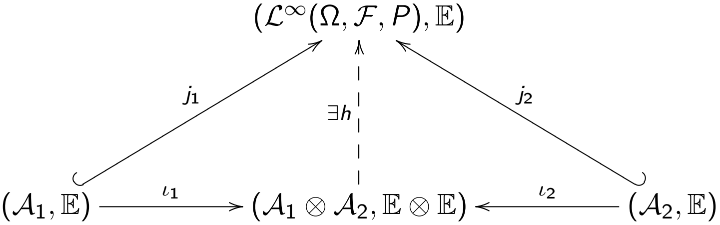

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and, by necessity,

$$h\big(f(\mathtt{X}_1) \otimes g(\mathtt{X}_2)\big) = f(\mathtt{X}_1) \cdot g(\mathtt{X}_2).$$

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Note then for f, g in $\mathcal{B}_b(\mathbb{R})$ that

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Thus the commutativity of the diagram is equivalent to the condition:

$$\forall f, g \in \mathcal{B}_b(\mathbb{R}) \colon \mathbb{E}\big[f(\mathtt{X}_1) \cdot g(\mathtt{X}_2)\big] = \mathbb{E}[f(\mathtt{X}_1)] \cdot \mathbb{E}[g(\mathtt{X}_2)].$$

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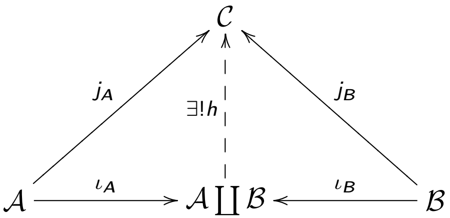

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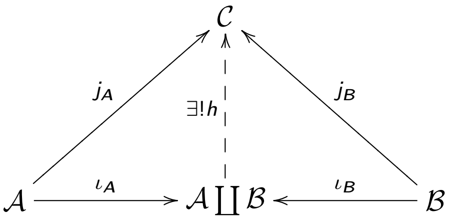

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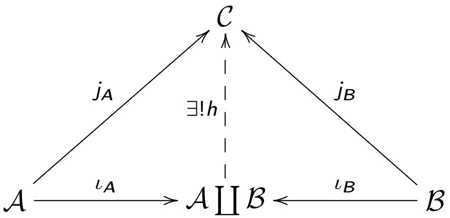

- ι_A and ι_B are canonical embeddings.
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In this case the product "$\bullet$" is determined by the operation:

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(\mathcal{A}_1, \varphi|_{\mathcal{A}_1}) & \xrightarrow{\iota_1} & (\mathcal{A}_1 \coprod \mathcal{A}_2, \varphi|_{\mathcal{A}_1} \bullet \varphi|_{\mathcal{A}_2}) & \xleftarrow{\iota_2} & (\mathcal{A}_2, \varphi|_{\mathcal{A}_2})
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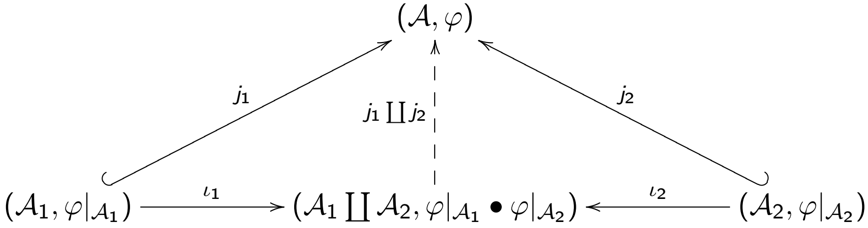

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Independence thus amounts to the condition:

$$\varphi \circ (j_1 \coprod j_2) = \varphi|_{\mathcal{A}_1} \bullet \varphi|_{\mathcal{A}_2}.$$

Natural conditions to impose on "$\bullet$"

Consider pairs

$$(\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2), (\mathcal{A}_3, \varphi_3), (\mathcal{C}_1, \psi_1), (\mathcal{C}_2, \psi_2)$$

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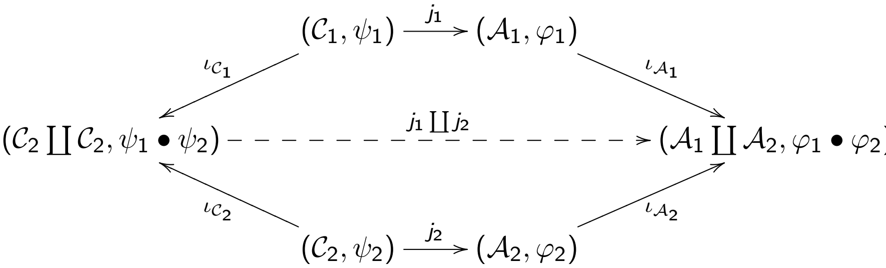

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Definitions of the 5 universal products

Let $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ be pairs from $\mathfrak{AwF}$.

Let further $a_1, a_2, \ldots, a_n$ be elements from $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{A}_1 \coprod \mathcal{A}_2$, such that

$$a_i \in \mathcal{A}_1 \Rightarrow a_{i+1} \in \mathcal{A}_2, \quad \text{and} \quad a_i \in \mathcal{A}_2 \Rightarrow a_{i+1} \in \mathcal{A}_1$$

for all i in $\{1, 2, \ldots, n-1\}$.

The set of elements of the form $a_1 a_2 \cdots a_n$ (subject to the conditions above) generate $\mathcal{A}_1 \coprod \mathcal{A}_2$.

Suppose for simplicity of notation that n is odd and that

$$a_1, a_3, a_5, \ldots, a_n \in \mathcal{A}_1, \quad \text{and} \quad a_2, a_4, a_6, \ldots, a_{n-1} \in \mathcal{A}_2.$$

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- The tensor product $\varphi_1 \otimes \varphi_2$ is defined by:

$$\varphi_1 \otimes \varphi_2(a_1 a_2 \cdots a_n) = \varphi_1(a_1 a_3 \cdots a_n)\varphi_2(a_2 a_4 \cdots a_{n-1}).$$

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$$\varphi_1 \star \varphi_2(a_1 a_2 \cdots a_n) = \sum_{I \subsetneq \{1,2,\ldots,n\}} \varphi \left(\overrightarrow{\prod_{i \in I}} a_i \right) \left(\prod_{i \notin I} \varphi_{\epsilon(i)}(a_i) \right)$$

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- The boolean product $\varphi_1 \diamond \varphi_2$ is defined by

$$\varphi \diamond \varphi_2(a_1 a_2 \cdots a_n) = \prod_{i=1}^{n} \varphi_{\epsilon(i)}(a_i)$$

$$= \varphi_1(a_1)\varphi_2(a_2)\varphi_1(a_3) \cdots \varphi_2(a_{n-1})\varphi_1(a_n).$$

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- The monotone product $\varphi_1 \triangleright \varphi_2$ is defined by

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- The anti-monotone product $\varphi_1 \triangleleft \varphi_2$ is defined by

$$\varphi_1 \triangleleft \varphi_2(a_1 a_2 \cdots a_n) = \left(\prod_{i:\ \epsilon(i)=1} \varphi_1(a_i) \right) \varphi_2 \left(\overrightarrow{\prod_{i:\ \epsilon(i)=2}} a_i \right)$$

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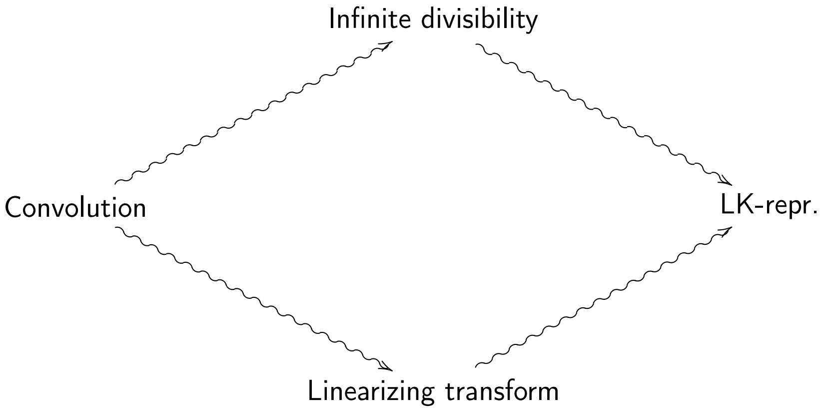
General answer: Represent a and b as selfadjoint operators on a Hilbert space. Then $\mu \Box \nu$ is the spectral distribution of the selfadjoint operator $a + b$.

Associated notion of infinite divisibility



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and

$$m_k(\mu) = \sum_{p=1}^{k} \sum_{\substack{i_1,\dots,i_p\geq 1 \\ i_1+\cdots+i_p=k}} r_{i_1}(\mu)r_{i_2}(\mu)\cdots r_{i_p}(\mu), \qquad (k\geq 2).$$

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Equivalently,

$$m_k(\mu) = \sum_{i=1}^{k} r_i(\mu) m_{k-i}(\mu), \qquad (k\in\mathbb{N}).$$

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We then have

$$r_k(\mu \Box \nu) = r_k(\mu) + r_k(\nu), \qquad (k \in \mathbb{N}).$$

The Boolean cumulant transform

Suppose μ and ν are (compactly supported) probability measures on $\mathbb{R}$, and consider the Laurent series:

$$G_\mu(z) = \sum_{n=0}^{\infty} m_n(\mu) z^{-n-1} = \int_{\mathbb{R}} \frac{1}{t - z}\, \mu(\mathrm{d}t),$$

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(i) $K_\mu(z) = z - \dfrac{1}{G_\mu(z)}$.

(ii) $K_{\mu \boxdot \nu}(z) = K_\mu(z) + K_\nu(z)$.

Proof of (i)

Note that

$$G(z)K(z) = \Big(\sum_{n=0}^{\infty} m_n(\mu) z^{-n-1}\Big)\Big(\sum_{k=1}^{\infty} r_k(\mu) z^{-k+1}\Big)$$

$$= \sum_{\ell=1}^{\infty}\Big(\sum_{k=1}^{\ell} r_k(\mu) m_{l-k}(\mu)\Big) z^{-\ell}$$

$$= \sum_{\ell=1}^{\infty} m_\ell(\mu) z^{-\ell}$$

$$= zG_\mu(z) - 1.$$

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For a function $K: \mathbb{C}^+ \to \mathbb{C}$ the following conditions are equivalent:

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$$K(z) = a + \int_{\mathbb{R}} \frac{1 + tz}{z - t}\,\tau(\mathrm{d}t), \qquad (z \in \mathbb{C}^+).$$

All probability measures on $\mathbb{R}$ are infinitely divisible w.r.t. Boolean convolution!

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Then for any z in $\mathbb{C}^+$

$$K_{\underbrace{\mu_n \Box \cdots \Box \mu_n}_{n \text{ terms}}}(z) = \sum_{j=1}^{n} K_{\mu_n}(z) = nK_{\mu_n}(z) = K_\mu(z).$$

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By uniqueness of Cauchy transforms, this means that

$$\mu_n \Box \cdots \Box \mu_n = \mu.$$

The Boolean Central Limit Theorem

Let μ be a probability measure on $\mathbb{R}$ with mean 0 and variance σ^2, and for each n in $\mathbb{N}$, let μ_n be the measure defined by

$$\mu_n(B) = \mu(\sigma\sqrt{n}B), \qquad (B \in \mathcal{B}(\mathbb{R})).$$

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Then

$$\mu_n \Box \cdots \Box \mu_n \xrightarrow{\text{w}} \frac{1}{2}(\delta_{-1} + \delta_1), \quad \text{as } n \to \infty.$$

Classical and free Lévy-Khintchine representation

A probability measure μ on $\mathbb{R}$ is infinitely-divisible w.r.t. classical convolution, if and only if there exists a finite measure σ and a real constant γ such that

$$\log f_\mu(u) = \gamma + \int_{\mathbb{R}} \left(\mathrm{e}^{iut} - 1 - \frac{iut}{1+t^2} \right) \frac{1+t^2}{t^2} \sigma(\mathrm{d}t), \qquad (u \in \mathbb{R}).$$

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A probability measure μ on $\mathbb{R}$ is infinitely-divisible w.r.t. free convolution, if and only if there exists a finite measure σ and a real constant γ such that

$$zG_\mu^{\langle -1\rangle}(z) - 1 = \gamma z + \int_{\mathbb{R}} \frac{z^2 + tz}{1 - tz}\,\sigma(\mathrm{d}t).$$