Convergence of the weighted quadratic variations of some fractional Brownian sheets

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Let $f : \mathbb{R} \to \mathbb{R}$ a regular deterministic function, $W := (W_{(s,t)})_{(s,t)\in[0,1]^2}$ a two-parameter stochastic process and $G_n := \left\{ \left(\frac{i}{n}, \frac{j}{n}\right), \ 1 \leq i,j \leq n \right\}$ a regular grid whith mesh $1/n$. In this talk we study the asymptotic behavior as n tends to infinity of the following quantity

$$\sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} f\left(W_{\left(\frac{i-1}{n},\frac{j-1}{n}\right)}\right) |\Delta_{i,j} W|^2 \quad n \geq 1, \ (s,t) \in [0,1]^2$$

where

$$\Delta_{i,j} W := W_{\left(\frac{i-1}{n},\frac{j-1}{n}\right)} + W_{\left(\frac{i}{n},\frac{j}{n}\right)} - W_{\left(\frac{i-1}{n},\frac{j}{n}\right)} - W_{\left(\frac{i}{n},\frac{j-1}{n}\right)}.$$

We are interested in two situations:

- W is a standard Brownian sheet,
- W is a fractional Brownian sheet.

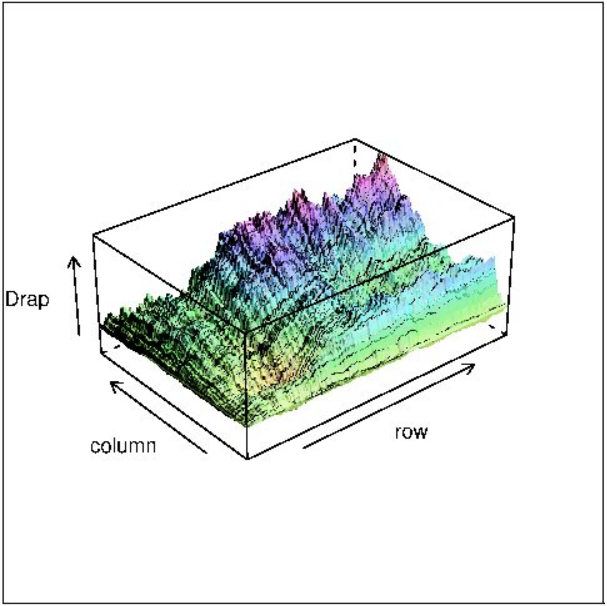
Let $W = \left(W_{(s,t)}\right)_{(s,t)\in[0,1]^2}$ be a two-parameter Brownian motion that is,

- $W_{(s,t)} = 0$, if $s = 0$ or $t = 0$,
- W is a centered Gaussian process with covariance function

$$\mathbb{E}\left[W_{(s,t)} W_{(s',t')}\right] = (s \wedge s')(t \wedge t'), \quad (s,t),(s',t') \in [0,1]^2.$$

Remark that $\Delta_{i,j} W$ and $\Delta_{i',j'} W$ are two independent rv if $(i,j) \neq (i',j')$.

The standard Brownian sheet



Let $f : \mathbb{R} \to \mathbb{R}$ a regular enough deterministic function. Let $X_n = (X_n(s,t))_{(s,t)\in[0,1]^2}$ defined as,

$$X_n(s,t) := n \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} f\left(W_{\left(\frac{i-1}{n},\frac{j-1}{n}\right)}\right)\left(|\Delta_{i,j}W|^2 - \frac{1}{n^2}\right), \quad n \geq 1.$$

Theorem (R.)

Under some reularity assumption on f we have that

$$X_n(\cdot,\bullet) := n \sum_{i=1}^{[n\cdot]} \sum_{j=1}^{[n\bullet]} f\left(W_{\left(\frac{i-1}{n},\frac{j-1}{n}\right)}\right)\left(|\Delta_{i,j}W|^2 - \frac{1}{n^2}\right) \xrightarrow[n\to\infty]{law(\mathcal{S})} X,$$

in the Skorohod space $\mathcal{D}([0,1]^2)$ and X is a non-Gaussian continuous process defined as

$$X_{(s,t)} := \sqrt{2}\int_{[0,s]\times[0,t]} f(W_\rho)\, dB_\rho.$$

The previous theorem has been established for the standard Brownian motion by Jacod, Gradinaru and Nourdin, Aït-Sahalia and Jacod, ...

Let

$$Y_{(s,t)} = \int_{[0,s]\times[0,t]} \sigma(W_\rho) dW_\rho + \int_{[0,s]\times[0,t]} M_\rho d\rho,$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ and M is a continuous two-parameter process.

We observe an unique path of Y on $\left\{ \left(\frac{i}{n}, \frac{j}{n} \right), \ 1 \leq i, j, \leq n \right\}$ with $n \geq 1$. Set

$$\begin{cases} V^n_{(s,t)} := \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \left| \Delta_{i,j} Y \right|^2, & (s,t) \in [0,1]^2, \ n \geq 1, \\ C_{(s,t)} := \int_{[0,s]\times[0,t]} \sigma^2 \left(W_{(u,v)} \right) \ dudv, & (s,t) \in [0,1]^2, \\ Y^n_{(s,t)} := n \left(V^n_{(s,t)} - C_{(s,t)} \right), & (s,t) \in [0,1]^2. \end{cases}$$

Let

$$V_n(s,t) := \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_{i,j} Y|^2, \quad (s,t) \in [0,1]^2.$$

Proposition

The estimator V_n of C is consistent:

$$V_n(s,t) \xrightarrow[n\to\infty]{L^2} C(s,t), \quad (s,t) \in [0,1]^2.$$

Rate of convergence this consitent estimator?

Proposition

Under some regularity assumptions on $\sigma(\cdot)$, the sequence on two-parameter processes

$$Y_n(\cdot,\bullet) := n\,(V_n - C)(\cdot,\bullet) \xrightarrow[n\to\infty]{law(\mathcal{S})} \sqrt{2} \int_{[0,\cdot]\times[0,\bullet]} \sigma^2(W_\rho)\, dB_\rho$$

in the Skorohod space $D([0,1]^2)$.

Theorem

Let,

$$S_n(s,t) := n^2 \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_{i,j} Y|^4, \quad S(s,t) := 3 \int_{[0,s]\times[0,t]} \sigma^4(W_\rho)\, d\rho.$$

We have for every (s,t) *in* $(0,1]^2$ *that*

$$S_n(s,t)^{-1/2} Y_n(s,t) \xrightarrow[n\to\infty]{law(\mathcal{S})} \sqrt{\frac{2}{3}} N, \quad N \sim \mathcal{N}(0,1).$$

Lemme (Aldous and Eagleson)

If $(Y_n)_n$ *converges stably in law to* Y *and if* $(S_n)_n$ *converges in probability to* S *then,*

$$(Y_n, S_n)_n \xrightarrow{law} (Y, S).$$

Let $B^H := (B^H_t)_{t\in[0,1]}$ be a fractional Brownian motion with Hurst index $0 < H < 1$.
Given a deterministic function $f : \mathbb{R} \to \mathbb{R}$ regular enough we are interested in the asymptotic behavior of the following quantity as n tends to infinity,

$$V_n(f) := \sum_{k=0}^{n-1} f(B^H_{k/n}) \left[n^{2H} (B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}})^2 - 1 \right]$$

The non-weighted case ($f(\cdot) = 1$) has been investigated by Breuer and Major, Dobrushin and Major, Giraitis and Surgailis, Taqqu who have shown that the value $H = 3/4$ is a particular value in this study.

Theorem (Nourdin, Nourdin and Nualart, Nourdin and Nualart and Tudor)

- *If* $H \in (0, 1/4)$ *then*

$$n^{2H-1} V_n(f) \xrightarrow[n \to \infty]{L^2} \frac{1}{4} \int_0^1 f''(B_s^H) ds,$$

- *if* $1/4 < H < 3/4$ *then*

$$n^{-1/2} V_n(f) \xrightarrow[n \to \infty]{law(\mathcal{S})} C_H \int_0^1 f(B_s^H) dW_s,$$

W standard Brownian motion independent of B^H,

- *if* $H = 3/4$ *then*

$$(n \log(n))^{-1/2} V_n(f) \xrightarrow[n \to \infty]{law(\mathcal{S})} C_{3/4} \int_0^1 f(B_s^{3/4}) dW_s,$$

W standard Brownian motion independent of $B^{1/4}$,

- *if* $H > 3/4$ *then*

$$n^{1-2H} V_n(f) \xrightarrow[n \to \infty]{L^2} \int_0^1 f(B_s^H) dZ_s, \text{ where } Z \text{ is a Rosenblatt process.}$$

The case $H = 1/4$ is not contained in the previous result.

Theorem (Nourdin, R.)

$$\begin{aligned} n^{-1/2} V_n(f) &= n^{-1/2} \sum_{k=0}^{n-1} f(B^{1/4}_{k/n}) \left[\sqrt{n} (B^{1/4}_{\frac{k+1}{n}} - B^{1/4}_{\frac{k}{n}})^2 - 1 \right] \\ &\xrightarrow[n\to\infty]{law(\mathcal{S})} C_{1/4} \int_0^1 f(B_s^{1/4}) dW_s + \frac{1}{4} \int_0^1 f''(B_s^{1/4}) ds, \end{aligned}$$

W standard Brownian motion independent of B^H.

For $f : \mathbb{R} \to \mathbb{R}$ regular enough we aim at defining

$$\int_0^1 f(B_s^{1/4}) \circ dB_s^{1/4} := \lim_{n\to\infty} \begin{cases} S_n(f) \\ T_n(f) \end{cases}$$

where

$$S_n(f) = \sum_{k=0}^{n-1} \frac{f(B_{k/n}^{1/4}) + f(B_{(k+1)/n}^{1/4})}{2} \left(B_{(k+1)/n}^{1/4} - B_{k/n}^{1/4}\right),$$

$$T_n(f) = \sum_{k=1}^{\lfloor n/2 \rfloor} f(B_{(2k-1)/n}^{1/4}) \left(B_{(2k)/n}^{1/4} - B_{(2k-2)/n}^{1/4}\right).$$

Theorem (Gradinaru, Nourdin, Russo, Vallois; Cheridito, Nualart)

It holds that

$$\int_0^1 f'(B_s^{1/4})d^\circ B_s^{1/4} := \lim_{n\to\infty} S_n(f') \quad \textit{exists in probability}$$

with

$$\int_0^1 f'(B_s^{1/4})d^\circ B_s^{1/4} = f(B_1^{1/4}) - f(0).$$

Theorem (Nourdin, R.; Burdzy, Swanson)

$$\int_0^1 f'(B_s^{1/4})d^\star B_s^{1/4} := \lim_{n\to\infty} T_n(f') \quad \textit{exists in law}$$

and satisfy

$$\int_0^1 f'(B_s^{1/4})d^\star B_s^{1/4} \overset{\text{law}}{=} f(B_1^{1/4}) - f(0) - \frac{\kappa}{2}\int_0^1 f''(B_s^{1/4})dW_s$$

where κ denotes an explicit constant and W is a standard BM independent of $B^{1/4}$.

Theorem (Nourdin, R., Swanson)

$$\int_0^1 f'(B_s^{1/6})d^\circ B_s^{1/6} := \lim_{n\to\infty} S_n(f') \quad \textit{exists in law}$$

and satisfy

$$\int_0^1 f'(B_s^{1/6})d^\circ B_s^{1/6} \overset{\text{law}}{=} f(B_1^{1/6}) - f(0) - \frac{\kappa_{1/6}}{2}\int_0^1 f'''(B_s^{1/6})dW_s$$

where $\kappa_{1/6}$ *denotes an explicit constant and* W *is a standard BM independent of* $B^{1/6}$.

Definition (Ayache, Léger and Pontier)

A fractional Brownian sheet $(B^{\alpha,\beta}_{(s,t)})_{(s,t)\in[0,1]^2}$ with Hurst indices $(\alpha,\beta)\in(0,1)^2$ is a centered two-parameter Gaussian process equal with

$$\{(s,t)\in[0,1]^2,\ s=0 \text{ or } t=0\}$$

whose covariance function is given by,

$$\begin{aligned} R^{\alpha,\beta}((s_1,t_1),(s_2,t_2)) &:= \mathbb{E}\left[B^{\alpha,\beta}_{(s_1,t_1)}B^{\alpha,\beta}_{(s_2,t_2)}\right] \\ &= K^{\alpha}(s_1,s_2)K^{\beta}(t_1,t_2) \\ &= \frac{1}{2}\left(s_1^{2\alpha}+s_2^{2\alpha}-|s_1-s_2|^{2\alpha}\right)\frac{1}{2}\left(t_1^{2\beta}+t_2^{2\beta}-|t_1-t_2|^{2\beta}\right). \end{aligned}$$

$(B^{\alpha,\beta}_{(s,t)})_{(s,t)\in[0,1]^2}$ is a semimartingale if and only if $(\alpha,\beta)=(1/2,1/2)$ and in this case this is the standard Brownian sheet studied previously.

Theorem (R.)

Let $f : \mathbb{R} \to \mathbb{R}$ a deterministic function regular enough. Let

$$X_n(s,t) := n^{-1} \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} f\left(B^{\alpha,\beta}_{\left(\frac{i-1}{n},\frac{j-1}{n}\right)}\right)\left(n^{2(\alpha+\beta)}|\Delta_{i,j}B^{\alpha,\beta}|^2 - 1\right),\ (s,t) \in [0,1]^2$$

be the re-normalized weighted quadratic variations where the increments $\Delta_{i,j}B^{\alpha,\beta}$ are defined as

$$\Delta_{i,j}B^{\alpha,\beta} := B^{\alpha,\beta}_{\left(\frac{i-1}{n},\frac{j-1}{n}\right)} + B^{\alpha,\beta}_{\left(\frac{i}{n},\frac{j}{n}\right)} - B^{\alpha,\beta}_{\left(\frac{i-1}{n},\frac{j}{n}\right)} - B^{\alpha,\beta}_{\left(\frac{i-1}{n},\frac{j}{n}\right)}.$$

If $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ with $\alpha + \beta > \frac{1}{2}$ then

$$X_n \xrightarrow[n\to\infty]{fdd-law(\mathcal{S})} X$$

with $X_{s,t} := \sigma_{\alpha,\beta} \int_{[0,s]\times[0,t]} f\left(B^{\alpha,\beta}_{(u,v)}\right) dW_{(u,v)},\ (s,t) \in [0,1]^2$ where W is a standard Brownian sheet independent of $B^{\alpha,\beta}$.