Multipower Variation

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Semimartingales and the observation scheme

- In this talk we consider Itô semimartingales, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \underbrace{(x1_{\{|x|\leq 1\}}) * (\mu_t - \nu_t)}_{\text{small jumps}} + \underbrace{(x1_{\{|x|>1\}}) * \mu_t}_{\text{big jumps}},$$

where W is a standard Brownian motion, a is a drift process, σ is the volatility, μ is a jump measure and ν is its predictable compensator.

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  where W is a standard Brownian motion, a is a drift process, σ is the volatility, μ is a jump measure and ν is its predictable compensator.

- We are in the context of high-frequency observations, i.e. the values

$$X_{i\Delta_n}\ , \qquad i = 1, \ldots, [t/\Delta_n]$$

  are observed, t is fixed and $\Delta_n \to 0$.

Identifiable objects: complete observation case

- Assume that we can observe the whole path $(X_s)_{s\in[0,t]}$ of the Itô semimartingale X. Then we can make the following observation:

  (i) We can identify the volatility process $(\sigma_s)_{s\in[0,t]}$.

  (ii) We can identify the jump part $(\Delta X_s)_{s\in[0,t]}$ of X ($\Delta X_s = X_s - X_{s-}$).

  (iii) We can identify the quadratic variation process $([X,X]_s)_{s\in[0,t]}$.

  (iv) We *can't* identify the drift process $(a_s)_{s\in[0,t]}$ (unless $\sigma \equiv 0$)!

  (v) We *can't* identify the law of the jump part of X (Levy case)!

  (vi) But we can identify the *activity* of jumps (cf. *Blumenthal-Getoor index*).

Standard statistical questions

- In practice, people are interested in obtaining information about the unobserved path of X from discrete observations $X_{i\Delta_n}$, $i = 1, \dots, [t/\Delta_n]$. Typical statistical problems are:
  (i) How to estimate the quadratic variation
  $$[X, X]_t = \int_0^t \sigma_s^2 ds + \sum_{s \le t} |\Delta X_s|^2 < \infty$$
  of X?
  (ii) How can we estimate functionals of σ (typically $\int_0^t |\sigma_s|^p ds$ for $p > 0$)?
  (iii) Does the unobserved path of X have jumps?
  (iv) Does the unobserved path of X contain a Brownian part?
  (v) What is the activity of the jump process?
  (vi) What is the "relative contribution" of various parts of X?

Various useful functionals

- To solve the afore-mentioned problems the following classes of functional are extremely useful: ($\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$):
  (i) Continuous case:

$$V(f)_t^n = \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) \qquad (\textit{Power variation})$$

$$V(f)_t^n = \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}, \ldots, \frac{\Delta_{i+k-1}^n X}{\sqrt{\Delta_n}}\Big) \qquad (\textit{Multipower variation})$$

  (ii) Discontinuous case:

$$\overline{V}(f)_t^n = \sum_{i=1}^{[t/\Delta_n]} f(\Delta_i^n X)$$

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  (ii) Discontinuous case:

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- The name *power variation* comes from the fact that we usually use $f(x) = |x|^p$.

Law of large numbers: continuous case, power variation

- Here we consider a continuous Itô semimartingale of the type

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \ ,$$

where a is predictable and locally bounded, and σ is càdlàg adapted. Define

$$\rho_x(f) = \mathbb{E}[f(xU)] \ , \qquad x \in \mathbb{R} \ , U \sim N(0,1).$$

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- **Theorem:** Assume that $f \in C_p(\mathbb{R})$. Then it holds

$$V(f)^n_t \xrightarrow{ucp} V(f)_t = \int_0^t \rho_{\sigma_s}(f) ds.$$

In the special case $f(x) = |x|^p$, $p > 0$, we obtain ($m_p = \mathbb{E}[|N(0,1)|^p]$)

$$V(f)^n_t \xrightarrow{ucp} m_p \int_0^t |\sigma_s|^p ds.$$

Law of large numbers: discontinuous case

- In the discontinuous case we only consider the functions $f(x) = |x|^p$, $p > 0$. Recall that $\overline{V}(f)^n_t = \sum_{i=1}^{[t/\Delta_n]} f(\Delta_i^n X)$.

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- **Theorem (Lepingle (1976)):** For *all* semimartingales we obtain the convergence

$$\overline{V}(f)^n_t \xrightarrow{\mathbb{P}} \begin{cases} [X,X]_t = \int_0^t \sigma_s^2 ds + \sum_{s\le t} |\Delta X_s|^2 & \text{for } p=2 \\ \sum_{s\le t} |\Delta X_s|^p & \text{for } p>2 \end{cases}$$

with $\Delta X_s = X_s - X_{s-}$.

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  with $\Delta X_s = X_s - X_{s-}$.

- Roughly speaking, the jump part dominates for powers $p > 2$ whereas the continuous part dominates for powers $0 < p < 2$. In the following we will see that in the continuous case we require a certain normalization to obtain non-trivial limits.

Discontinuous case: robust estimation I

- For various statistical problems we need to estimate certain characteristics of the continuous part in the presence of the jump part. One idea is to use a *threshold-based estimator* proposed Mancini (2004):

$$TRV(X,\varpi)_t^n = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 1_{\{|\Delta_i^n X| \leq c\Delta_n^{\varpi}\}},$$

  where $\varpi \in (0, 1/2)$.

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- **Theorem:** Let $\varpi \in (0, 1/2)$. For Itô semimartingales we obtain the convergence

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- A similar result holds for a truncated version of $V(f)_t^n$ with $f(x) = |x|^p$ for powers $0 < p < 2$. In the case $p > 2$ we require further assumptions on the activity of the jump part to deduce robustness.

Discontinuous case: robust estimation II

- Another idea of obtaining jump robust measures for *all* powers of volatility is the multipower variation of the form

$$V(X, p_1, \dots p_k, \Delta_n)_t = \Delta_n^{1-\frac{p^+}{2}} \sum_{i=1}^{[t/\Delta_n]-k+1} |\Delta_i^n X|^{p_1} \cdots |\Delta_{i+k-1}^n X|^{p_k}$$

where $p_j \geq 0$ and $p^+ = \sum p_j$. This concept goes back to Barndorff-Nielsen and Shephard.

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- **Theorem:** If $\max_j(p_j) < 2$ and X is an Itô semimartingales, it holds that

$$V(X, p_1, \dots p_k, \Delta_n)_t \xrightarrow{ucp} m_{p_1} \cdots m_{p_k} \int_0^t |\sigma_s|^{p^+} ds$$

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- Indeed, this class provides jump robust estimates for all positive powers p: choose $k \in \mathbb{N}$ with $p/k < 2$ and use the powers $p_j = p/k$, $j = 1, \dots, k$.

Application: estimation of the jump quadratic variation

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$$V(X,2,\Delta_n)_t - TRV(X,\varpi)_t^n \xrightarrow{\mathbb{P}} \sum_{s\leq t} |\Delta X_s|^2.$$

- *Multipower approach:*

$$V(X,2,\Delta_n)_t - m_1^{-2} V(X,1,1,\Delta_n)_t \xrightarrow{\mathbb{P}} \sum_{s\leq t} |\Delta X_s|^2.$$

Application: Test for jumps I

- Barndorff-Nielsen & Shephard (2004) use the multipower variation method to construct a test for jumps. They define two test statistics:

$$S_t^{n,1} = \Delta_n^{-1/2}(V(X,2,\Delta_n)_t - m_1^{-2} V(X,1,1,\Delta_n)_t)$$

$$S_t^{n,2} = \Delta_n^{-1/2}\Big(\frac{m_1^{-2} V(X,1,1,\Delta_n)_t}{V(X,2,\Delta_n)_t} - 1\Big)$$

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- Large values of $S_t^{n,1}$ indicate the presence of jumps; negative values of $S_t^{n,2}$ ($<< 0$) also speak for a substantial influence of the jump part.

- Remark: Positive values of $S_t^{n,2}$ can be interpreted as an indication that X is not an Itô semimartingale.

Application: Test for jumps II

- Ait-Sahalia & Jacod (2008) apply the asymptotic theory to test for the presence of jumps. They use the "change of frequency" approach:

$$S_t^n = \frac{V(X,4,2\Delta_n)_t}{V(X,4,\Delta_n)_t}.$$

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- Our LLN results imply that

$$S_t^n \xrightarrow{\mathbb{P}} \begin{cases} 2: & \text{when X has a continuous path} \\ 1: & \text{when X has a discontinuous path} \end{cases}$$

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- The derivation of a formal test procedure is straightforward once we obtain a central limit theorem for the statistic S_n (still to come).

Definition of stable convergence

In this talk we will intensively use the concept of stable convergence which is due to Renyi (1963).

- **Definition:** A sequence Y_n on $(\Omega, \mathcal{F}, \mathbb{P})$ converges stably in law to the limit Y ($Y_n \xrightarrow{st} Y$), that is defined on the extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, iff for any real-valued function $g \in C_b$ and any bounded $\mathcal{F}$-measurable variable Z it holds that

$$\lim_{n\to\infty} \mathbb{E}[g(Y_n)Z] = \mathbb{E}'[g(Y)Z].$$

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  Clearly, stable convergence is stronger than weak convergence.

- Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. When the above convergence holds for any $\mathcal{G}$-measurable variable Z, then Y_n is said to converge $\mathcal{G}$-stably in law towards Y In this case we write

$$Y_n \xrightarrow{\mathcal{G}_{st}} Y.$$

Properties of stable convergence I

(i) *(General relation)* We have

$$Y_n \xrightarrow{\mathbb{P}} Y \quad \Longrightarrow \quad Y_n \xrightarrow{st} Y \quad \Longrightarrow \quad Y_n \xrightarrow{d} Y.$$

(ii) *(Alternative definition)* It holds that

$$Y_n \xrightarrow{st} Y \quad \Leftrightarrow \quad (Y_n, Z) \xrightarrow{d} (Y, Z) \quad \Leftrightarrow \quad (Y_n, Z) \xrightarrow{st} (Y, Z).$$

for any $\mathcal{F}$-measurable variable Z.

(iii) *(Joint convergence)* Let $Y_n \xrightarrow{st} Y$, $Z_n \xrightarrow{\mathbb{P}} Z$. Then

$$(Y_n, Z_n) \xrightarrow{st} (Y, Z).$$

.

Properties of stable convergence II

(iv) *(Why extension?)* Assume that $Y_n \xrightarrow{st} Y$ and Y is $\mathcal{F}$-measurable. Then

$$Y_n \xrightarrow{\mathbb{P}} Y.$$

(v) *(Stable Δ-method)* Let $\sqrt{n}(Y_n - Y) \xrightarrow{st} X$ and $g \in C^1$. Then

$$\sqrt{n}(g(Y_n) - g(Y)) \xrightarrow{st} g'(Y)X.$$

(vi) *(Crucial application)* Assume that $\sqrt{n}(Y_n - Y) \xrightarrow{st} VU$, where $U \sim N(0,1)$, $V > 0$ unknown $\mathcal{F}$-measurable rv with $V \perp U$ (*mixed normality*). If $V_n^2 \xrightarrow{\mathbb{P}} V^2$ it holds that

$$\frac{\sqrt{n}(Y_n - Y)}{V_n} \xrightarrow{st} U \sim N(0,1).$$

The nature of stable convergence

- Let us consider a sequence $(X_i)_{i\geq 1}$ of i.i.d. rv with $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$, defined on $(\Omega, \mathcal{F}, P)$. Assume that $\mathcal{F} = \sigma(X_1, X_2, \ldots)$. We obtain

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{d} N(0,1).$$

Is there a "stable version" of this CLT? Yes! Indeed, it is rather easy to prove that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{st} U \sim N(0,1) \ ,$$

where U is defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $U \perp \mathcal{F}$.

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where U is defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $U \perp \mathcal{F}$.

- In fact, this is a typical situation: we usually only require a "new" standard normal variable in the case of stable convergence for rv's, or a "new" Brownian motion in the case of stable convergence of processes.

CLT: The continuous case

- Here we consider a continuous Itô semimartingale X of the form

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- We assume additionally that the volatility process σ is itself an Itô semimartingale. The following result is established in Jacod (1994), Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) and Kinnebrock & Podolskij (2008).

The stable CLT

Theorem: Let $f \in C_p^1(\mathbb{R})$ be an *even* function. Then we obtain

$$\Delta_n^{-1/2}(V(f)_t^n - V(f)_t) \xrightarrow{st} L(f)_t = \int_0^t v_s dW_s' \ ,$$

where W' is a Brownian motion independent of $\mathcal{F}$, and

$$v_s^2 = \rho_{\sigma_s}(f^2) - \rho_{\sigma_s}^2(f).$$

Moreover, if $f(x) = |x|^p$ with $p > 0$, it holds that

$$\Delta_n^{-1/2}(V(f)_t^n - V(f)_t) \xrightarrow{st} \sqrt{m_{2p} - m_p^2} \int_0^t |\sigma_s|^p dW_s',$$

where $m_p = \mathbb{E}[|N(0,1)|^p]$.

A feasible CLT

- Notice that the limit process $L(f)_t = \int_0^t v_s dW'_s$ is *mixed normal* with mean 0 and conditional variance $\int_0^t v_s^2 ds$ ($L(f)_t = MN(0, \int_0^t v_s^2 ds)$). In this case we can always obtain a standard CLT by the properties of stable convergence.

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- *Example:* Consider the function $f(x) = |x|^p$ with $p > 0$. Recall that in this case the conditional variance is given as

$$\int_0^t v_s^2 ds = (m_{2p} - m_p^2) \int_0^t |\sigma_s|^{2p} ds.$$

Consequently, it holds that

$$\frac{\Delta_n^{-1/2}(V(f)_t^n - V(f)_t)}{\sqrt{\frac{m_{2p} - m_p^2}{m_{2p}} V(f^2)_t^n}} \xrightarrow{d} N(0,1).$$

Estimation of the conditional variance: general case

- Recall again that

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- The following statistic is the most natural estimator of the conditional variance $\int_0^t v_s^2 ds$:

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \left(f^2\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) f\Big(\frac{\Delta_{i+1}^n X}{\sqrt{\Delta_n}}\Big)\right)$$

$$\xrightarrow{ucp} \int_0^t v_s^2 ds.$$

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$$\xrightarrow{ucp} \int_0^t v_s^2 ds.$$

- Note: *natural* does not mean *optimal*!

Idea of the proof

- First, note the approximation

$$\frac{\Delta_i^n X}{\sqrt{\Delta_n}} = \Delta_n^{-1/2}\Big(\underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} a_s ds}_{=O_{\mathbb{P}}(\Delta_n)} + \underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dW_s}_{=O_{\mathbb{P}}(\Delta_n^{1/2})}\Big)$$

$$\approx \Delta_n^{-1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W = \alpha_i^n.$$

Idea of the proof

- First, note the approximation

$$\frac{\Delta_i^n X}{\sqrt{\Delta_n}} = \Delta_n^{-1/2}\Big(\underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} a_s ds}_{=O_{\mathbb{P}}(\Delta_n)} + \underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dW_s}_{=O_{\mathbb{P}}(\Delta_n^{1/2})}\Big)$$

$$\approx \Delta_n^{-1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W = \alpha_i^n.$$

- In the next step we set $\chi_i^n = \Delta_n^{1/2}\Big(f(\alpha_i^n) - \mathbb{E}[f(\alpha_i^n)|\mathbb{F}_{(i-1)\Delta_n}]\Big)$ and prove that

$$L(f)_t^n = \sum_{i=1}^{[t/\Delta_n]} \chi_i^n \xrightarrow{st} L(f)_t.$$

$L(f)_t^n \xrightarrow{st} L(f)_t$

The following result follows from Jacod (1997).

- **Main Theorem:** When

  (i) $\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_i^n|^2 | \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} F_t = \int_0^t v_s^2 ds,$

  (ii) $\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_i^n \Delta_i^n W | \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0,$

  (iii) $\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[\chi_i^n \Delta_i^n N | \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0$ for all bounded N with $[W, N] \equiv 0,$

  (iv) $\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[|\chi_i^n|^2 1_{\{|\chi_i^n| > \varepsilon\}} | \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0$ for all $\varepsilon > 0,$

  then we obtain

$$L(f)_t^n = \sum_{i=1}^{[t/\Delta_n]} \chi_i^n \xrightarrow{st} L(f)_t = \int_0^t v_s dW_s'.$$

Application: Approximation of solutions of SDE's

- Let X be a continuous Itô semimartingale and Y is a strong solution of the SDE

$$Y_t = Y_0 + \int_0^t f(Y_s)dX_s \ , \qquad f \in C^1(\mathbb{R}).$$

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- Let Y^n be an Euler approximation of this solution, i.e.

$$dY_t^n = f(Y_{\phi_n(t)}^n) dX_t \ , \qquad Y_0^n = Y_0 \ , \qquad \phi_n(t) = \Delta_n [t/\Delta_n]$$

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- We are interested in the asymptotic behaviour of the approximation error

$$U_t^n = Y_t^n - Y_t.$$

Set

$$Z_t^n(X) = \int_0^t (X_s - X_{\Delta_n[s/\Delta_n]})dX_s.$$

Application: Approximation of solutions of SDE's

The following result goes back to Jacod & Protter (1998).

Proposition: It holds that

$$\Delta_n^{-1/2} U_t^n \xrightarrow{st} U_t \quad \Leftrightarrow \quad \Delta_n^{-1/2} Z_t^n \xrightarrow{st} Z_t.$$

In this case U is a known functional of X and Z, i.e. $U = F(X, Z)$.

Proposition: An application of Itô's formula shows that

$$\Delta_n^{-1/2}\Big(V(X,2,\Delta_n)_t - [X,X]_t\Big) = 2\Delta_n^{-1/2} \int_0^{\Delta_n[t/\Delta_n]} (X_s - X_{\Delta_n[s/\Delta_n]}) dX_s.$$

We immediately deduce that

$$\Delta_n^{-1/2} Z_t^n \xrightarrow{st} Z_t = \frac{1}{\sqrt{2}} \int_0^t \sigma_s^2 dW_s'.$$

CLT: the discontinuous case

- Now we consider Itô semimartingales of the type

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + (x1_{\{|x|\leq 1\}}) * (\mu_t - \nu_t) + (x1_{\{|x|>1\}}) * \mu_t.$$

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- In this case we only consider functions of the form $f(x) = |x|^p$, $p \geq 2$. Recall the law of large numbers

$$\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 \xrightarrow{\mathbb{P}} [X,X]_t = \int_0^t \sigma_s^2 ds + \sum_{s\leq t} |\Delta X_s|^2 ,$$

$$\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p \xrightarrow{\mathbb{P}} \sum_{s\leq t} |\Delta X_s|^p \qquad \text{for } p > 2.$$

In the next step we will show the associated stable CLT's (see Jacod (2008)).

Stable CLT's

- For $f(x) = |x|^p$ $(p \geq 2)$ let us introduce the following process:

$$\overline{L}(f)_t = \sum_{T_m \leq t} f'(\Delta X_{T_m}) \Big(\sqrt{\kappa_m} \sigma_{T_m -} U_m + \sqrt{1 - \kappa_m} \sigma_{T_m} U'_m \Big).$$

Here $(T_m)_{m \geq 1}$ denotes the jump times of X, $(U_m)_{m \geq 1}$ and $(U'_m)_{m \geq 1}$ are i.i.d. $N(0,1)$ and $(\kappa_m)_{m \geq 1}$ are i.i.d. $\mathcal{U}([0,1])$.

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- The processes $(U_m)_{m\geq 1}$, $(U'_m)_{m\geq 1}$, $(\kappa_m)_{m\geq 1}$ and $(W'_t)_{t\geq 0}$ are all defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$, are mutually independent and independent of $\mathcal{F}$.

Stable CLT's

- **Theorem:** Let $f(x) = |x|^p$, $p \geq 2$. We obtain the following results:

  (i) For $p > 3$ and any fixed $t > 0$ it holds that

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p - \sum_{s\leq t} |\Delta X_s|^p\Big) \xrightarrow{st} \overline{L}(f)_t.$$

  (ii) For $p = 2$ and any fixed $t > 0$ it holds that

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 - [X,X]_t\Big) \xrightarrow{st} L(f)_t + \overline{L}(f)_t.$$

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- Note that for $p \in (2,3]$ the CLT is not available. Furthermore, notice that the above CLT's *never* hold in a functional sense when jumps are present.

A feasible CLT

- For simplicity we consider the case $p > 3$. Assume that X and σ have *no common* jumps. Then, for any $t > 0$,

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p - \sum_{s\le t} |\Delta X_s|^p\Big) \xrightarrow{st} MN\Big(p^2 \sum_{T_m \le t} |\Delta X_{T_m}|^{2(p-1)} \sigma_{T_m}^2\Big).$$

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- Now, the conditional variance can be estimated as follows:

$$p^2 \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^{2(p-1)}\hat{\sigma}_{i\Delta_n}^2 \xrightarrow{\mathbb{P}} p^2 \sum_{T_m\leq t} |\Delta X_{T_m}|^{2(p-1)}\sigma_{T_m}^2$$

where $\hat{\sigma}_{i\Delta_n}^2 = h_n^{-1}(TRV(X,\varpi)_{i\Delta_n+h_n}^n - TRV(X,\varpi)_{i\Delta_n}^n)$ with $h_n \to 0$ and $h_n/\Delta_n \to \infty$ (see Ait-Sahalia & Jacod (2008) and Veraart (2008)).

Alternative method

- Consider again the case $p > 3$. Here we drop the assumption that X and σ have no common jumps. In this case the limit process $\overline{L}(f)_t$ is *not* mixed normal.

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- However, we may obtain a feasible CLT as follows: for any $l = 1, \ldots, M$ generate

$$\overline{L}(f)_t^{n,l} = \sum_{i=1}^{[t/\Delta_n]} f'(\Delta_i^n X) \left(\sqrt{\kappa_m^{(l)}} \hat{\sigma}_{i\Delta_n -} U_m^{(l)} + \sqrt{1 - \kappa_m^{(l)}} \hat{\sigma}_{i\Delta_n} U_m'^{(l)} \right).$$

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- *Conjecture:* It holds that $\overline{L}(f)_t^{n,l} \xrightarrow{d} \overline{L}(f)_t$ as $n, M \to \infty$.

Idea of the proof: the case $p = 2$

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- The proof is performed by showing that

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^c|^2 - \int_0^t \sigma_s^2 ds\Big) \xrightarrow{st} L(f)_t,$$

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^d|^2 - \sum_{s\leq t} |\Delta X_s|^2\Big) \xrightarrow{\mathbb{P}} 0,$$

$$2\Delta_n^{-1/2} \sum_{i=1}^{[t/\Delta_n]} \Delta_i^n X^d \Delta_i^n X^c \xrightarrow{st} \overline{L}(f)_t.$$

An assumption on the jump activity

- A transformation argument implies that the jump part X^d of X has the form

$$X_t^d = \int_0^t \int_{\mathbb{R}} \kappa \circ \delta(s,x)(\overline{\mu} - \overline{\nu})(ds,dx) + \int_0^t \int_{\mathbb{R}} \kappa' \circ \delta(s,x)\overline{\mu}(ds,dx),$$

where $\overline{\mu}$ is a Poisson random measure with compensator $\overline{\nu}(ds,dx) = ds \otimes dx$, κ is a truncation function and $\kappa'(x) = x - \kappa(x)$.

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- **(L-q):** We assume that δ is càglàd and there exists a sequence of stopping times $S_k \nearrow \infty$ and a sequence of functions $(\gamma_k(x))_{k\geq 1}$ s.t. for $s \leq T_k$

$$|\delta(s,x)| \leq \gamma_k(x) \qquad \text{and} \qquad \int_{\mathbb{R}} (1 \wedge \gamma_k^q(x))dx < \infty$$

for some $q \in [0,2]$.

Robust CLT I: threshold estimator

- Recall the convergence:

$$TRV(X,\varpi)_t^n = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 1_{\{|\Delta_i^n X| \leq c\Delta_n^{\varpi}\}} \xrightarrow{ucp} \int_0^t \sigma_s^2 ds$$

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with $\varpi \in (0, 1/2)$.

- **Theorem:** Assume that X is a discontinuous Itô semimartingale and (L-q) holds with $q < \frac{4\varpi}{2\varpi - 1}$ (it implies that $\varpi > 1/4$ and $q < 1$). Then

$$\Delta_n^{-1/2}\Big(TRV(X,\varpi)_t^n - \int_0^t \sigma_s^2 ds\Big) \xrightarrow{st} L(f)_t = \sqrt{2}\int_0^t \sigma_s^2 dW_s',$$

where $f(x) = x^2$ and $L(f)_t$ is the limit process in the continuous case.

Intuition behind the proof

- Let X^d denote the discontinuous part of X. It is sufficient to prove that

$$\Delta_n^{-1/2} TRV(X^d, \varpi)_t^n = \Delta_n^{-1/2} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^d|^2 1_{\{|\Delta_i^n X^d| \leq c\Delta_n^{\varpi}\}} \xrightarrow{\mathbb{P}} 0.$$

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- For any $\delta > 0$ small, it holds that

$$\begin{aligned} \Delta_n^{-1/2} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^d|^2 1_{\{|\Delta_i^n X^d| \le c\Delta_n^{\varpi}\}} &\le \Delta_n^{-1/2+\varpi(2-q-\delta)} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^d|^{q+\delta} \\ &\sim \Delta_n^{-1/2+\varpi(2-q-\delta)} \sum_{s\le t} |\Delta X_s^d|^{q+\delta} \end{aligned}$$

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- If $\delta > 0$ is small enough the latter converges to 0 in probability, because $q < \frac{4\varpi}{2\varpi-1}$.

Robust CLT II: multipower estimator

- Recall the law of large numbers ($p_j \geq 0$ and $p^+ = \sum p_j$):

$$V(X, p_1, \dots p_k, \Delta_n)_t = \Delta_n^{1-\frac{p^+}{2}} \sum_{i=1}^{[t/\Delta_n]-k+1} \prod_{l=1}^{k} |\Delta_{i+l-1}^n X|^{p_l}$$

$$\xrightarrow{ucp} m_{p_1} \cdots m_{p_k} \int_0^t |\sigma_s|^{p^+} ds.$$

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- **Theorem:** Assume that X is a discontinuous Itô semimartingale and (L-q) holds with $\frac{q}{2-q} < p_j < 1$. Then

$$\Delta_n^{-1/2}\left(V(X, p_1, \dots p_k, \Delta_n)_t - m_{p_1} \cdots m_{p_k} \int_0^t |\sigma_s|^{p^+} ds \right) \xrightarrow{st} A \int_0^t |\sigma_s|^{p^+} dW'_s,$$

  where the constant A depends on $p_1, \dots, p_k$ and the limit process remains the same in the continuous case. That is, the CLT is robust to jumps.

Intuition behind the proof

- Let us consider the case $k = 1$, $p_1 = p$ and assume that the jump part of X is a q-stable process S^q. Obviously, it suffices to show that

$$\Delta_n^{-1/2} V(S^q, p, \Delta_n)_t = \Delta_n^{\frac{1-p}{2}} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n S^q|^p \xrightarrow{\mathbb{P}} 0.$$

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- *Case $p > q$:* The latter clearly holds, because $p < 1$ and

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- *Case $p < q$:* Due to the self-similarity of S^q we deduce

$$\Delta_n^{\frac{1-p}{2}} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n S^q|^p \sim \Delta_n^{-\frac{1+p}{2} + \frac{p}{q}} \mathbb{E}[|S_1^q|^p],$$

which converges to 0 as $\frac{q}{2-q} < p$.

Thank you!