Stochastic Integrals and Conditional Full Support

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2. for all $t \in [0, T)$ and $\mathbf{P}$-almost all $\omega \in \Omega$,

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- For price processes $I = \mathbb{R}_+$, otherwise $I = \mathbb{R}$. Conventionally $\mathbb{F} = \mathbb{F}^X$ (the usual augmentation).

Why CFS is worth studying

Pricing models with transaction costs

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- More precisely, GRS have shown that if price process $(P_t)_{t\in[0,T]}$ has CFS, then the superreplication price of a European (vanilla) contingent claim $g(P_T)$ under ε-sized proportional transaction costs tends to

$$\hat{g}(P_0) \quad \text{when } \varepsilon \downarrow 0,$$

where $\hat{g}$ is the *concave envelope* of g.

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Other

- *Riemann integrals* of processes with CFS (GRS).

Some characterizations of CFS

Proposition (Small-ball probabilities)

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$$\mathbf{P}\left[\sup_{u\in[t,T]} |X_u - X_t - f(u)| < \varepsilon \middle| \mathscr{F}_t\right] > 0 \quad \mathbf{P}\text{-}a.s.$$

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for all $t \in [0, T)$, $f \in C_0([t, T], \mathbb{R})$, and $\varepsilon > 0$.

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- Intuitively, this characterization means that for every $t \in [0, T)$, $f \in C_0([t, T], \mathbb{R})$, $\varepsilon > 0$, and for almost every "past", the following event occurs with a positive $\mathscr{F}_t$-conditional probability:

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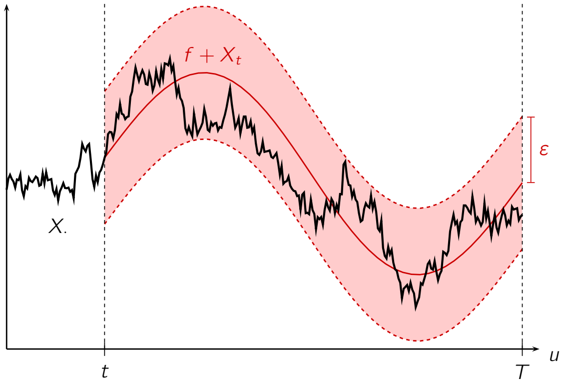

Proposition (Usual augmentation)

Again, let $(X_t)_{t\in[0,T]}$ *be a continuous process in* $I \subset \mathbb{R}$*, adapted to filtration* $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$.

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Proposition (Law invariance)

Let $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ be continuous processes in $I \subset \mathbb{R}$, such that $X \overset{\text{law}}{=} Y$. Then, X has CFS w.r.t. $\mathbb{F}^X$ if and only if Y has CFS w.r.t. $\mathbb{F}^Y$.

Main results: CFS for certain stochastic integrals

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Let us define

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Application to some stochastic volatility models

General stochastic volatility (SV) model

Let us consider price process $(P_t)_{t\in[0,T]}$ in $\mathbb{R}_+$ given by

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Since W is independent of B and V, the previous Theorem implies that $\log P$ has CFS—from which it follows that P has CFS (when P is seen as a process in $\mathbb{R}_+$).

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- V is a non-semimartingale (Comte–Renault [long memory in volatility]),
- V is discontinuous (Barndorff-Nielsen–Shephard [subordinator-driven volatility], Guo [regime switching]).

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Theorem (Progressive integrands and Brownian integrators)

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- *and* $\mathscr{F}_t := \{\xi, Y_s, W_s : s\in[0,t]\}$.

If W *is an* $(\mathscr{F}_t)_{t\in[0,T]}$*-Brownian motion and*

$$|h(\cdot)| \le \overline{h}, \quad \overline{k}^{-1} \le |k(\cdot)| \le \overline{k} \quad \text{for some } \overline{h} > 0 \text{ and } \overline{k} > 1,$$

Main results: CFS for certain stochastic integrals

Theorem (Progressive integrands and Brownian integrators)

Suppose that

- $(Y_t)_{t\in[0,T]}$ *and* $(W_t)_{t\in[0,T]}$ *are continuous processes,*
- h *and* k *are progressive maps* $[0,T]\times C([0,T],\mathbb{R})^2 \to \mathbb{R}$*,*
- ξ *is a random variable,*
- *and* $\mathscr{F}_t := \{\xi, Y_s, W_s : s\in[0,t]\}$*.*

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then

$$Z_t := \xi + \int_0^t h(s,Y,W)\mathrm{d}s + \int_0^t k(s,Y,W)\mathrm{d}W_s,\quad t\in[0,T]$$

has CFS.

Application to stochastic differential equations

Weak solutions of stochastic differential equations

Let us consider price process $(P_t)_{t\in[0,T]}$ in $\mathbb{R}_+$ given by

$$\mathrm{d}P_t = \mu(t,P)\mathrm{d}t + \sigma(t,P)\mathrm{d}W_t, \quad P_0 = p_0 \in \mathbb{R}_+,$$

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- there exist $\overline{\mu} > 0$ and $\overline{\sigma} > 1$ such that

$$|\mu(t,x)| \le \overline{\mu}x(t), \quad \overline{\sigma}^{-1}x(t) \le |\sigma(t,x)| \le \overline{\sigma}x(t)$$

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Setting $Y := P$, we find that the previous Theorem applies to $\log P$, and hence that P has CFS (similarly as with the SV model).

Main results: CFS for certain stochastic integrals

Theorem (Independent integrands and general integrators)

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References

- C. Bender, T. Sottinen, E. Valkeila, Pricing by hedging and no-arbitrage beyond semimartingales, Finance Stoch. 12 (4) (2008) 441–468.
- A. Cherny, Brownian moving averages have conditional full support, Ann. Appl. Probab. 18 (5) (2008) 1825–1830.
- D. Gasbarra, T. Sottinen, H. van Zanten, Conditional full support of Gaussian processes with stationary increments, Preprint 487, Department of Mathematics and Statistics, University of Helsinki (2008).
- P. Guasoni, M. Rásonyi, W. Schachermayer, Consistent price systems and face-lifting pricing under transaction costs, Ann. Appl. Probab. 18 (2) (2008) 491–520.
- M. S. Pakkanen, Stochastic integrals and conditional full support, arXiv:0811.1847 (2009).
- D. W. Stroock, S. R. S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, in: Proc. Sixth Berkeley Symp. on Math. Statist. Probab. III: Probability theory, Univ. California Press, Berkeley, Calif., 1972.