Increment martingales

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Starting point

- Some interesting *martingale-like* processes are not local martingales:
  - Lévy processes indexed by $\mathbb{R}$ (that is, they have stationary independent increments) with centered increments;
  - Diffusions (index set $(0, +\infty)$) on natural scale started at $+\infty$ (i.e. $X_0 := \lim_{s \to 0} X_s = +\infty$) which is an entrance boundary.
- At the same time one would like to define an integral with respect to such processes.
- The general theory of martingales indexed by, say, partially ordered sets, does not seem to give much insight in this case.
- We develop a framework in which the above processes can be analysed.

Content

- Martingales indexed by $\mathbb{R}$.
- Increment martingales:
  - $[\cdot]$ and $\langle\cdot\rangle$;
  - Martingales are increment martingales. Conversely: Increment (local) martingale plus ... implies (local) martingale up to addition of random variables.
- Integration
- Extensions.

INDEX SET: $\mathbb{R}$.

Martingales indexed by $\mathbb{R}$

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, P)$.

- $M = (M_t)_{t\in\mathbb{R}}$ is a martingale if it is adapted, integrable and $E[M_t|\mathcal{F}_s] = M_s$ for all $s < t$.
  In this case: $M_{-\infty}$ exists a.s. and $(M_t)_{t\in[-\infty,\infty)}$ is a martingale.

- $M = (M_t)_{t\in\mathbb{R}}$ is a local martingale if there is a localising sequence $(\tau_n)_{n\geq 1}$ such that M^{τ_n} is a martingale.
  In this case: $M_{-\infty}$ exists a.s. and $(M_t)_{t\in[-\infty,\infty)}$ is a local martingale.

In particular: Lévy processes indexed by $\mathbb{R}$ cannot be local martingales.

Increments

Let $X = (X_t)_{t\in\mathbb{R}}$ be a real-valued process. For $s \in \mathbb{R}$ let

$$ {}^sX_t := X_t - X_{t\wedge s} = \begin{cases} 0 & \text{if } t \le s \\ X_t - X_s & \text{if } t \ge s. \end{cases} $$

(The increment over the interval $(s, t]$.)
Set furthermore ${}^sX = ({}^sX_t)_{t\in\mathbb{R}}$.

Increment martingales - Definition

- $X = (X_t)_{t\in\mathbb{R}}$ is an *increment martingale* if for all s, ${}^sX \in \mathcal{M}$. That is, ${}^sX$ is an adapted process and

$$E[{}^sX_t|\mathcal{F}_s] = 0 \quad \text{for all } s < t.$$

- $X = (X_t)_{t\in\mathbb{R}}$ is an *increment local martingale* if for all s ${}^sX \in \mathcal{LM}$.
  In this case the localising sequence may depend on s.

Note: Increment local martingales are in general not adapted or integrable. In fact:

- M increment local martingale and Z random variable implies $M + Z$ is increment local martingale.

Increment martingales - remarks and examples

- (Local) martingale implies increment (local) martingale.
- A Lévy process indexed by $\mathbb{R}$ with centered increments is an increment martingale with respect to the filtration generated by increments. That is,

$$\mathcal{F}_t = \sigma({}^sX_u : s \leq u \leq t).$$

- A diffusion on natural scale with ∞ as entrance boundary is an increment local martingale. (Stretch index set $(0, \infty)$ into $(-\infty, \infty)$.)

Increment (local) martingale plus . . . implies (local) martingale

Increment (local) martingale plus ... implies (local) martingale

- Increment martingale plus adapted plus integrable equals martingale.
- Assume M is in $\mathcal{IM}$ and $({}^sM_0)_{s<0}$ is UI.
  Then: $M_{-\infty}$ exists a.s. and $M - M_{-\infty}$ is in $\mathcal{M}$.
- Assume M is in $\mathcal{IM}^2$ and $\sup_{s:s\leq 0} E[({}^sM_0)^2] < \infty$.
  Then: $M_{-\infty}$ exists a.s. and $M - M_{-\infty}$ is in $\mathcal{M}^2$.

Existence of $M_{-\infty}$ alone does not imply that $M - M_{-\infty}$ is in $\mathcal{M}$.

Example

Assume: $\tau_1 \sim \tau_2 \sim f \sim F$ (independent) where $F(t) \in (0,1)$ for all t.

Let $N^i_t = 1_{[\tau_i,\infty)}(t)$, $N_t = (N^1_t, N^2_t)$ and let the filtration be generated by N.

Let

$$X_t = (\tau_1 \wedge \tau_2 \wedge t)(N^1_{\tau_1\wedge\tau_2\wedge t} - N^2_{\tau_1\wedge\tau_2\wedge t}) = \int_{-\infty}^{\tau_1\wedge\tau_2\wedge t} s\,\mathrm{d}(N^1 - N^2)_s.$$

Note that X is an adapted step function and $X_{-\infty} = 0$.

If

$$\int_{-\infty}^{t} \frac{uf(u)}{1-F(u)}\,\mathrm{d}u = -\infty \quad \text{for all } t$$

then X is in $\mathcal{IM}$ but not in $\mathcal{LM}$.

Increment (local) martingale plus ... implies (local) martingale

Assume $M \in \mathcal{ILM}^2$. Then the following are equivalent

(1) $M_{-\infty}$ exists a.s. and $M - M_{-\infty}$ is in $\mathcal{LM}^2$.

(2) There exists a predictable increasing process $\langle M \rangle$ with $\langle M \rangle_{-\infty} = 0$ such that for all s, ${}^s\langle M \rangle$ is the predictable quadratic variation for ${}^sM$, i.e. ${}^s\langle M \rangle = \langle {}^sM \rangle$.

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Let $M \in \mathcal{ILM}^2$. There exists a generalised predictable quadratic variation for ${}^sM$, to be denoted $\langle M \rangle^{\mathrm{g}}$. This process is unique up to addition of random variables and satisfies ${}^s\langle M \rangle^{\mathrm{g}} = \langle {}^sM \rangle$ for all s. Thus, (2) can be rephrased as: $\lim_{s \to -\infty} \langle M \rangle^{\mathrm{g}}_s$ is finite.

Increment (local) martingale plus ... implies (local) martingale

Assume $M \in \mathcal{ILM}$ is continuous.
If $M_{-\infty}$ exists a.s., then $M - M_{-\infty}$ is in $\mathcal{LM}$.

Note: It its not enough that $M_{-\infty}$ exists in probability.

Increment (local) martingale plus . . . implies (local) martingale

Assume $M \in \mathcal{ILM}$.
The existence of a quadratic variation $[M]$ *does not* imply that $M_{-\infty}$ exists a.s., and $M - M_{-\infty}$ is in $\mathcal{LM}$. See previous example.

$[M]$ is an increasing process with $[M]_{-\infty} = 0$ such that for all s ${}^{s}[M] = [{}^{s}M]$.

Increment (local) martingale plus ... implies (local) martingale

But: $[M]$ exists iff there is a continuous local martingale component of M and $\sum_{s<0}(\Delta M_s)^2 < \infty$.
The first condition is that M can be decomposed as

$$M = M^c + M^d$$

where M^c is a continuous local martingale and for all s, ${}^sM^d$ is a purely discontinuous local martingale.

Moreover: $[M]$ exists and $[M]^{1/2}$ locally integrable iff $M_{-\infty}$ exists a.s. and $M - M_{-\infty}$ is in $\mathcal{LM}$.

Integration - improper integrals

Let $M \in \mathcal{ILM}$.
The integral $(\int_s^t \phi_u \,\mathrm{d}M_u)_{t \geq s}$ is then well-defined.
Fix t. If $\lim_{s \to -\infty} \int_s^t \phi_u \,\mathrm{d}M_u$ exists a.s.: Improper integral.
(Existence of improper integrals does not depend on t).

- Improper integrals are in general not local martingales up to addition of random variables except when M is continuous.
- Necessary and sufficient conditions for existence of improper integrals?

Integration - proper integrals

Let $M \in \mathcal{ILM}$.
One can define a proper integral $\phi \bullet M_t = \int_{-\infty}^t \phi_u \, \mathrm{d}M_u$.

- This integral is a local martingale and $\phi \bullet M_{-\infty} = 0$
- Existence of $\phi \bullet M$ implies existence of the improper integral.
- When M is continuous the proper integral exists iff the improper integral exists. In fact, these integrals exist iff

$$\int_{-\infty}^t \phi_u^2 \, \mathrm{d}\langle M \rangle_u^{\mathrm{g}} < \infty.$$

Extensions - to appear

Keywords:

- Increment semimartingales
- Vector-valued random measures
- Integration with respect to Vector-valued random measures/ISM.