Are fractional Brownian motions predictable?

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Preliminaries

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$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ is a stochastic basis, satisfying the "usual" conditions, i.e. the filtration $\{\mathcal{F}_t\}$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}_T$. By convention $\mathcal{F}_\infty = \mathcal{F}$.

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Let $\{X_t\}_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$, adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$ (i.e. for each $t \in [0, T]$, X_t is $\mathcal{F}_t$ measurable) and with regular (or càdlàg) trajectories (i.e. its P-almost all trajectories are right-continuous and possess limits from the left on $(0, T]$).

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Suppose we are sampling the process $\{X_t\}$ at points $0 = t_0^\theta < t_1^\theta < t_2^\theta < \ldots < t_{k^\theta}^\theta = T\}$ of a partition θ of the interval $[0, T]$. By the discretization of X on θ we mean the process

$$X^\theta(t) = X_{t_k^\theta} \quad \text{if} \quad t_k^\theta \leq t < t_{k+1}^\theta, \; X_T^\theta = X_T.$$

Preliminaries - continued

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Suppose random variables $\{X_t\}_{t\in[0,T]}$ are integrable. We associate with any discretization X^θ its "predictable compensator"

$$
\begin{aligned}
A_t^\theta &= 0 \quad \text{if} \quad 0 \leq t < t_1^\theta, \\
A_t^\theta &= \sum_{j=1}^{k} E\left(X_{t_j^\theta} - X_{t_{j-1}^\theta} \middle| \mathcal{F}_{t_{j-1}^\theta}\right) \quad \text{if} \quad t_k^\theta \leq t < t_{k+1}^\theta,\ k \leq k^\theta - 1, \\
A_T^\theta &= \sum_{j=1}^{k^\theta} E\left(X_{t_j^\theta} - X_{t_{j-1}^\theta} \middle| \mathcal{F}_{t_{j-1}^\theta}\right).
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is a martingale with respect to the discrete filtration $\{\mathcal{F}_t\}_{t\in\theta}$.
- Suppose $EX_t^2 < +\infty, \ t \in [0, T]$. Fix θ and let $\mathcal{A}^\theta$ be the set of discrete-time stochastic processes $\{A_t\}_{t\in\theta}$ which are $\{\mathcal{F}_t\}_{t\in\theta}$-predictable.

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- A^θ_t is $\mathcal{F}_{t^\theta_{k-1}}$-measurable for $t^\theta_k \leq t < t^\theta_{k+1}$ (predictability!).
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- Suppose $EX_t^2 < +\infty,\ t \in [0,T]$. Fix θ and let $\mathcal{A}^\theta$ be the set of discrete-time stochastic processes $\{A_t\}_{t\in\theta}$ which are $\{\mathcal{F}_t\}_{t\in\theta}$-predictable. Then the predictable compensator $\{A^\theta_t\}_{t\in\theta}$ minimizes the functional

$$\mathcal{A}^\theta \ni A \mapsto E[X - A]_T,$$

  where the discrete quadratic variation $[\cdot]$ is defined as usual by

$$[Y]_T = \sum_{t\in\theta} (\Delta Y_t)^2 = \sum_{k=1}^{k^\theta} (Y_{t^\theta_k} - Y_{t^\theta_{k-1}})^2.$$

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Definition

We will say that an adapted stochastic process $\{X_t\}_{t\in[0,T]}$ with regular trajectories admits a local predictor $\{C_t\}_{t\in[0,T]}$ along $\Theta = \{\theta_n\}$ and in the sense of convergence $\to_\tau$ if

$$A^{\theta_n} \to_\tau C$$

and C has regular trajectories.

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Any continuous and nondecreasing adapted integrable process coincides with its local predictor in the sense of the uniform convergence in probability.

Submartingales of class D

Any submartingale of class D with continuous increasing process in the Doob-Meyer decomposition admits the local predictor which coincides with its predictable continuous compensator.

Fractional Brownian motions

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A fractional Brownian motion (fBm) $\{B_t^H\}_{t\in\mathbb{R}^+}$ of Hurst index $H \in (0,1)$ is a continuous and centered Gaussian process with covariance function

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

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Theorem

For $H \in (1/2, 1)$ the fractional Brownian motion $\{B_t^H\}_{t\in[0,T]}$ coincides with its local predictor along any sequence of normally condensing partitions and in the sense of the uniform convergence in probability.

Proof of the theorem

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$\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration generated by the fBm $\{B_t^H\}$. Let $\{\theta_n\}$ be a sequence of normally condensing partitions of $[0, T]$ and let $\{A_t^{\theta_n}\}_{t\in\theta_n}$ be the predictable compensator for the discretization of $\{(B^H)_t^{\theta_n}\}$ on θ_n.

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\begin{aligned}
E \sup_{t\in\theta_n}((B^H)_t^{\theta_n} - A_t^{\theta_n})^2 &\leq 4E(B_T^H - A_T^{\theta_n})^2 = 4E[(B^H)^{\theta_n} - A^{\theta_n}]_T \\
&\leq 4E[(B^H)^{\theta_n}]_T = 4\sum_{k=1}^{k^{\theta_n}} |t_k^{\theta_n} - t_{k-1}^{\theta_n}|^{2H} \\
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Since we have also almost surely

$$\sup_{t\in[0,T]} |(B^H)_t^{\theta_n} - B_t^H| \to 0,$$

the theorem follows.

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The above result is a direct consequence of the fact that for $H \in (1/2.1)$ the fBm is a process of energy zero in the sense of Fukushima, i.e.

$$E[X^{\theta_n}]_T = E \sum_{k=1}^{k^{\theta_n}} (X_{t_k^{\theta_n}} - X_{t_{k-1}^{\theta_n}})^2 \to 0, \quad \text{as } n \to \infty.$$

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$$\sum_{\{k\,:\,t_{k+1}^{\theta_n} \leq t\}} E((X_{t_{k+1}^{\theta_n}} - X_{t_k^{\theta_n}})^2 | \mathcal{F}_{t_k^{\theta_n}}) - (E(X_{t_{k+1}^{\theta_n}} - X_{t_k^{\theta_n}} | \mathcal{F}_{t_k^{\theta_n}}))^2 \to_P 0.$$

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For $H \in (0, 1/2)$ the fractional Brownian motion $\{B_t^H\}_{t\in[0,T]}$ admits no local predictor. In fact, for any sequence $\{\theta_n\}$ of normal condensing partitions we have

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Remark. The random variables $A_T^{\theta_n}$ are Gaussian, so $\sup_n E(A_T^{\theta_n})^2 = +\infty$ is equivalent to the lack of tightness of the family $\{A_T^{\theta_n}\}$. Thus in the case $H \in (0, 1/2)$ the compensators do not stabilize in any reasonable probabilistic sense.

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Lemma (Theorem of Nuzman and Poor (2000), with corrections of Ahn and Inoue (2004))

If $H \in (0, 1/2)$ then for $0 \leq s < t$ there exists a nonnegative function $h_{t,s}(u)$ such that

$$\int_0^s h_{t,s}(u)\, du = 1,$$

and

$$E(B_t^H | \mathcal{F}_s) = \int_0^s h_{t,s}(u) B_u^H du, \text{ a.s.}$$

(We work with the natural filtration $\mathcal{F}_s = \sigma\{B_u^H \, : \, 0 \leq u \leq s\}$.)

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Remark. It is possible to write down the exact (and complicated) form of the function $h_{t,s}$, but we do not need it.

The proof - continued

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Lemma

For $H \in (0, 1/2)$ and $0 \leq s < t$

$$E(B_t^H - E(B_t^H|\mathcal{F}_s))^2 = E(B_t^H - B_s^H - E(B_t^H - B_s^H|\mathcal{F}_s))^2 \geq \frac{1}{2}|t-s|^{2H}.$$

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Now the theorem follows immediately:

$$\begin{aligned} \sup_n E(B_T^H - A_T^{\theta_n})^2 &= \sup_n E[(B^H)^{\theta_n} - A^{\theta_n}]_T \\ &\geq \frac{1}{2}\sum_{k=1}^{k^{\theta_n}} |t_k^{\theta_n} - t_{k-1}^{\theta_n}|^{2H} \to +\infty. \end{aligned}$$

Submartingales - the general case

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If the compensator of a submartingale is discontinuous, we have then in general only weak in L^1 convergence of discrete compensators. Such convergence, although satisfactory from the analytical point of view, brings only little probabilistic understanding to the nature of the compensation.

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It was proved by A.J. (2005) that one may employ here the celebrated Komlós theorem on subsequences: given any sequence $\{\theta_n\}$ of partitions one can find a subsequence $\{n_j\}$ along which the Césaro means of compensators of discretizations converge to the limiting compensator.

Submartingales - the general case (continued)

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More precisely, if $\{n_j\}$ is the selected subsequence and we denote by $\{A^j_t\}$ the predictable compensator of the discretization on θ_{n_j}, then for each rational $t \in [0, T]$

$$B^N_t = \frac{1}{N} \sum_{j=1}^{N} A^j_t \to A_t, \quad \text{a.s.},$$

where A is the continuous-time process in the Doob-Meyer decomposition.

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$$B^N_t = \frac{1}{N}\sum_{j=1}^{N} A^j_t \to A_t, \quad \text{a.s.},$$

where A is the continuous-time process in the Doob-Meyer decomposition. In fact the above convergence can be strengthened: for each stopping time $\tau \leq T$ we have

$$\limsup_{N\to+\infty} B^N_\tau = A_\tau, \quad \text{a.s.}.$$

In particular, this directly implies predictability of $\{A_t\}$.

Processes with finite energy

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A.J. (2006) showed that the Komlós machinery works perfectly also for the Graversen-Rao decomposition: For a sequence $\{\theta_n\}$ of partitions of $[0, T]$ such that random variables $\{A_T^{\theta_n}\}$ are uniformly integrable one can select a subsequence such that for each stopping time $\tau \leq T$

$$B_\tau^N \to A_\tau, \quad \text{in } L^1.$$

Weak Dirichlet processes

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The process $\{X_t\}$ is weak Dirichlet if it admits a decomposition $X_t = M_t + A_t$, where $\{M_t\}$ is a local martingale, and $\{A_t\}$ is predictable and such that $[A, N] = 0$ for each continuous martingale N.

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The following two results are proved in the latter paper.

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- the generalized quadratic covariation $[X, N]$ does exist for every locally square integrable martingale N.

In such a case the decomposition $X_t = M_t + A_t$ is *unique* and $\{A_t\}$ is the *predictable compensator* of $\{X_t\}$ given by the G-R Theorem.

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- In the "natural" decomposition $X_t = M_t + A_t$ given by the G-R Theorem, $\{M_t\}$ is a square integrable martingale and $\{A_t\}$ admits the (classical) quadratic variation $[A, A]$, which is also square integrable.

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- In the "natural" decomposition $X_t = M_t + A_t$ given by the G-R Theorem, $\{M_t\}$ is a square integrable martingale and $\{A_t\}$ admits the (classical) quadratic variation $[A, A]$, which is also square integrable.
- The "natural" decomposition is *minimal*: if $X_t = M'_t + A'_t$ is another decomposition into a local martingale $\{M'_t\}$ and a predictable process $\{A'_t\}$, then $[A', A']$ exists and

$$[A', A'] = [M - M', M - M'] + [A, A].$$