# New central limit theorems for functionals of Gaussian processes and their applications

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Central limit theorems of random variables

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# Introduction

Consider a sequence of Gaussian random variables $\{X_n, n \geq 1\}$ with $E(X_j) = 0$, $E(X_j^2) = 1$, $E(X_j X_k) = \rho(j,k)$, $j,k \geq 1$. Let $H(x)$ be a real value function such that, $E(H(X_1)) = 0$ and $E(H(X_1))^2 < \infty$. Then, a natural problem, is to find suitable conditions ensuring that

$$F_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} H(X_i) \underset{n \to \infty}{\to} N(0,1).$$

# Introduction

So far, the main way of solving this problem was to check if the moments of $F_n$ converged to the moments of the standard normal distribution, that is, to see if

$$\lim_{n\to\infty} E(F_n^p) \underset{n\to\infty}{\to} \begin{cases} (p-1)!!, \text{ if } p \text{ is even} \\ 0 \text{ if } p \text{ is odd.} \end{cases}$$

# Introduction

To prove this one expanded $H(x)$ in the form

$$H(x) = \sum_{j=1}^{\infty} c_j H_j(x),$$

where $H_j$ is the $j$th Hermite polynomial

$$H_j(x) = (-1)^n e^{x^2/2} \frac{d^j}{dx^j}(e^{-x^2/2}), \quad j \geq 1,$$

and by using the **diagram formula** to calculate the asymptotic moments of $F_n$.

# Introduction

## Diagram formula

Consider a set of vertices $\{(i,j), 1 \leq i \leq p, 1 \leq j \leq l_i\}$ and a diagram $G$ with the following properties

1. Edges may pass only if the first coordinates of the vertices are different.
2. Each vertex has one edge.

Let $\Gamma = \Gamma(l_1, ..., l_n)$ denote the set of all these diagrams and for $G \in \Gamma$ by $A(G)$ the set of edges $G$. Now for a $w \in G$, $w = ((i_1, j_1), (i_2, j_2))$, where $i_1 < i_2$, define the functions $d_1(w) = i_1, d_2(w) = i_2$. Then

$$E(\Pi_{i=1}^{p} H_{l_i}(X_i)) = \sum_{G \in \Gamma} \Pi_{w \in A(G)} \rho(d_1(w), d_2(w)).$$

As a particular case we have

$$E(H_n(X_1) H_m(X_2)) = \delta_{nm} m! \rho^m(1,2),$$

where $\delta_{nm}$ is the Kronecker symbol.

# Introduction

Since the work of Nualart and Peccatti in (2005) we know that it is sufficient to check the behaviour of the second and the fourth order moments of $(F_n)$ , even we can get equivalent conditions for the convergence of the fourth order moments that are easier to check. Moreover the random variables $F_n$ can be measurable with respect to a Gaussian process, not necessarily discrete. The theoretical framework to obtain these results is the so called The Malliavin Calculus.

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian subspace $\mathcal{H}_1$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero-mean Gaussian random variables. Let $\mathfrak{H}$ be a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and norm $|| \cdot ||_{\mathfrak{H}}$. We will assume that there is an isometry

$$\begin{aligned} W : \mathfrak{H} &\rightarrow \mathcal{H}_1 \\ h &\mapsto W(h) \end{aligned}$$

in the sense that

$$E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathfrak{H}}.$$

It is easy to see that this map has to be linear. $W$ is called an isonormal Gaussian process.

## Example

Let $\{e_i, i \geq 1\}$ be the canonical basis of $\mathbb{R}^{\mathbb{N}}$ with a scalar product $\langle e_i, e_j \rangle = \rho(i, j)$ consider $\mathfrak{H} = span\{e_i, i \geq 1\}$. Then $\{W(e_i), i \geq 1\}$ will be a sequence of centered Gaussian random variables with covariance function $\rho(\cdot, \cdot)$.

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## Example

Take $\langle \mathbf{1}_{[0,t]}(\cdot), \mathbf{1}_{[0,s]}(\cdot) \rangle = \rho(s, t)$, and $\mathfrak{H} = span\{\mathbf{1}_{[0,t]}(\cdot), 0 \leq t \leq T\}$ then $\big(W_t := W(\mathbf{1}_{[0,t]})\big)$is a centered Gaussian process with covariance function $\rho(\cdot, \cdot)$.

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## Example

If, in the previous example, we take $\rho(s,t) = s \wedge t$ then $\mathfrak{H} = L^2([0,T], dx)$ and $\left(W_t := W(\mathbf{1}_{[0,t]})\right)$ is a Brownian motion, moreover $W(h) = \int_0^T h_s dW_s$, the Wiener integral of the function $h$ with respect to the Brownian motion $(W_t)$.

### Example

$\mathfrak{H} = L^2(\mathbb{A}, \mathcal{A}, \mu)$ where $(\mathbb{A}, \mathcal{A})$ is a measurable space and $\mu$ is a $\sigma$-finite measure without atoms (i.e. for any $A \in \mathcal{A}$ such that $\mu(A) > 0$ there is $B \in \mathcal{A}$ such that $0 < \mu(B) < \mu(A)$).
The process $\{W(A) := W(\mathbf{1}_A), A \in \mathcal{A}, \mu(A) < \infty\}$ is called a Gaussian white noise with intensity $\mu$ on the space $(\mathbb{A}, \mathcal{A})$.
We can define a Wiener integral of a function $h \in \mathfrak{H}$ with respect to the process $(W(A))$ and we have that $W(h) = \int_{\mathbb{A}} h_s dW_s$. We can also construct, in a standard way, the multiple Wiener integral for functions in $L^2(\mathbb{A}^n, \mathcal{A}^n, \mu^n)$ and it can be seen that, if $h \in \mathfrak{H}$, $||h||_{\mathfrak{H}} = 1$, $H_n(W(h)) = \int_{\mathbb{A}^n} h(t_1)...h(t_n)dW_{t_1}...dW_{t_n}$.

For any $m \geq 2$, we denote by $\mathcal{H}_m$ the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $H_m(W(h))$, where $h \in \mathfrak{H}$, $||h||_{\mathfrak{H}} = 1$. It is called the $m$-th Wiener chaos. Then,

## Theorem

*Every random variable* $Y \in L^2(\Omega, \mathcal{G}, P)$*, where* $\mathcal{G}$ *is the* $\sigma$*-field generated by* $W$*, can be uniquely expanded as*

$$Y = E(Y) + \sum_{n=1}^{\infty} Y_n,$$

*where* $Y_n \in \mathcal{H}_n$.

Proof. If $Y \in L^2(\Omega, \mathcal{G}, P)$ is orthogonal to every $H_n(W(h))$, $h \in \mathfrak{H}$, $||h||_{\mathfrak{H}} = 1$ then $Y$ is orthogonal to $e^{\sum_{i=1}^n \lambda_i W(e_i)}$, $\lambda_i \in \mathbb{R}, i \geq 1$ and $(e_i)_{i\geq 1}$ an orthonormal basis of $\mathfrak{H}$. From here $E(Y|W(e_1), ...W(e_n)) = 0, a.s$ and since $E(Y|W(e_1), ...W(e_n))$ converges a.s. to $Y$, then $Y = 0, a.s.$. ∎

Suppose that $\mathfrak{H}$ is infinite-dimensional and let $\{e_i, i \geq 1\}$ be an orthonormal basis of $\mathfrak{H}$. Denote by $\Lambda$ the set of all sequences $a = (a_1, a_2, ...),\ a_i \in \mathbb{N}$, such that all the terms, except a finite number of them, vanish. For $a \in \Lambda$ we set $a! = \Pi_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multiindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

The family of random variables $\{\Phi_a, a \in \Lambda\}$ is an orthonormal system. In fact

$$E\left[\Pi_{i=1}^{\infty} H_{a_i}(W(e_i)) \Pi_{i=1}^{\infty} H_{b_i}(W(e_i))\right] = \delta_{ab} a!\ ,$$

Moreover, $\{\Phi_a |\ a \in \Lambda,\ |a| = m\}$ is a complete orthonormal system in $\mathcal{H}_m$ .

Let $a \in \Lambda$ with $|a| = m$ and denote $\otimes_{i=1}^{\infty} e_i^{\otimes a_i} = e^{\otimes a}$. Where $\otimes$ is the tensor product. The mapping

$$\begin{aligned} I_m : \mathfrak{H}^{\odot m} &\rightarrow \mathcal{H}_m \\ \widetilde{e^{\otimes a}} &\mapsto \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)), \end{aligned}$$

between the symmetric tensor product $\mathfrak{H}^{\odot m}$, equipped with the norm $\sqrt{m!}\,\|\cdot\|_{\mathfrak{H}^{\otimes m}}$, and the $m$-th chaos $\mathcal{H}_m$ is a linear isometry. Here $\tilde{\otimes}$ denotes the symmetrization of the tensor product $\otimes$ and $I_0$ is the identity in $\mathbb{R}$.

For any $h = h_1 \otimes \cdots \otimes h_m$ and $g = g_1 \otimes \cdots \otimes g_m \in \mathfrak{H}^{\otimes m}$, we define the $p$-th contraction of $h$ and $g$, denoted by $h \otimes_p g$, as the element of $\mathfrak{H}^{\otimes 2(m-p)}$ given by

$$h \otimes_p g = \langle h_1, g_1 \rangle_{\mathfrak{H}} \cdots \langle h_p, g_p \rangle_{\mathfrak{H}} h_{p+1} \otimes \cdots \otimes h_m \otimes g_{p+1} \otimes \cdots \otimes g_m.$$

This definition can be extended by linearity to any element of $\mathfrak{H}^{\otimes m}$. $h \otimes_p g$ does not necessarily belong to $\mathfrak{H}^{\odot(2m-p)}$, even if $h$ and $g$ belong to $\mathfrak{H}^{\odot m}$. We denote by $h \widetilde{\otimes}_p g$ the symmetrization of $h \otimes_p g$.

## Proposition

*For any $h \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, we have*

$$I_p(h)I_q(g) = \sum_{r=0}^{p \wedge q} r! \begin{pmatrix} p \\ r \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} I_{p+q-2r}(h \widetilde{\otimes}_r g).$$

Proof. First, note that

$$I_1(e_i) = W(e_i).$$

Let $a \in \Lambda$ with $|a| = p$ and $q = 1$. Due to linearity of $I_p$ it suffices to consider the case $h = \widetilde{e^{\otimes a}}$, $g = e_j$. It holds that

$$I_p(\widetilde{e^{\otimes a}}) I_1(e_j) = \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)) W(e_j).$$

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Assume that $j$ is an index such that $a_j = 0$. Then

$$\widetilde{e^{\otimes a}} \widetilde{\otimes}_1 e_j = 0$$

and

# Basic Malliavin Calculus

## Multiplication Formula

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Assume that $j$ is an index such that $a_j = 0$. Then

$$\widetilde{e^{\otimes a}} \widetilde{\otimes}_1 e_j = 0$$

and

$$\Pi_{i=1}^{\infty} H_{a_i}(W(e_i)) W(e_j) = I_{p+1}(\widetilde{e^{\otimes a}} \widetilde{\otimes} e_j),$$

so we have that

Proof (cont.).

$$I_p(\widetilde{e^{\otimes a}})I_1(e_j) = I_{p+1}(\widetilde{e^{\otimes a}}\widetilde{\otimes} e_j) + pI_{p-1}(\widetilde{e^{\otimes a}}\widetilde{\otimes}_1 e_j).$$

# Basic Malliavin Calculus

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Assume now that $a_j \neq 0$. Then we obtain the identity

$$\widetilde{e^{\otimes a}}\widetilde{\otimes}_1 e_j = \frac{a_j}{p}\widetilde{e^{\otimes a'(j)}}$$

with $a'_i(j) = a_i$ if $i \neq j$ and $a'_j(j) = a_j - 1$.

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with $a_i'(j) = a_i$ if $i \neq j$ and $a_j'(j) = a_j - 1$. Furthermore,since the Hermite polynomials verify

$$xH_n(x) = H_{n+1}(x) + nH_{n-1}(x).$$

we have that

$$\begin{aligned} & \Pi_{i=1}^{\infty} H_{a_i}(W(e_i))W(e_j) \\ = \quad & \Pi_{i=1,i\neq j}^{\infty} H_{a_i}(W(e_i)(H_{a_j+1}(W(e_j)) + a_j H_{a_j-1}(W(e_j))) \end{aligned}$$

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Hence, the multiplication formula is true for $q = 1$. The general formula follows by induction. ■

### Theorem

*Let* $h \in \mathfrak{H}$ *with* $||h||_{\mathfrak{H}} = 1$. *Then for every* $m \geq 1$ *we have*

$$I_m(h^{\otimes m}) = H_m(W(h))$$

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$$I_{m+1}(h^{\otimes(m+1)}) = I_m(h^{\otimes m})I_1(h) - mI_{m-1}(h^{\otimes(m-1)})$$

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$$\begin{aligned} I_{m+1}(h^{\otimes(m+1)}) &= I_m(h^{\otimes m})I_1(h) - mI_{m-1}(h^{\otimes(m-1)}) \\ &= H_m(W(h)W(h) - mH_{m-1}(W(h)) \end{aligned}$$

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∎

### Theorem

*Every random variable* $Y \in L^2(\Omega, \mathcal{G}, P)$, *where* $\mathcal{G}$ *is the* $\sigma$*-field generated by* $W$, *can be uniquely expanded as*

$$Y = \sum_{n=0}^{\infty} I_n(h_n),$$

*where* $h_n \in \mathfrak{H}^{\odot n}$.

Proof. It is immediate from the previous chaos decomposition and the definition of $I_m$. ■

# Basic Malliavin Calculus

## The Malliavin derivative

Let $\mathcal{S}$ be the class of smooth random variables $F = f(W(h_1), ..., W(h_n))$, $f \in C_p^\infty(\mathbb{R}^n)$ ($f$ and all its partial derivatives have polynomial growth), we can define its differential as

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), W(h_2), ..., W(h_n))h_i.$$

$DF$ is as a random variable with values in $\mathfrak{H}$. Then we can built a closed map

$$\begin{array}{rcl} D : \mathbb{D}^{1,2} \subseteq L^2(\Omega, \mathbb{R}) & \longrightarrow & L^2(\Omega, \mathfrak{H}) \\ F & \mapsto & DF. \end{array}$$

where $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables with respect to the norm

$$||F||_{1,2} = \left( E(|F|^2) + E(||DF||^2_{\mathfrak{H}}) \right)^{1/2}.$$

For instance $D(H_n(W(h))) = nH_{n-1}(W(h))h, n \geq 1$, ($H_0 := 1$).

### Proposition

*Set* $h \in \mathfrak{H}^{\odot n}$, *then*

$$E(\|DI_n(h)\|_{\mathfrak{H}}^2) = nn! \, \|h\|_{\mathfrak{H}^{\otimes n}}^2$$

Proof. It is sufficient to consider $h = h_1^{\otimes n}$, $h_1 \in \mathfrak{H}$. Then,
$I_n(h_1^{\otimes n}) = ||h_1||_{\mathfrak{H}}^n H_n(W(h_1/||h_1||_{\mathfrak{H}}))$,

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$$DI_n(h) = n||h_1||_{\mathfrak{H}}^{n-1} H_{n-1}(W(h_1/||h_1||_{\mathfrak{H}}))h_1,$$

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$$DI_n(h) = n||h_1||_{\mathfrak{H}}^{n-1} H_{n-1}(W(h_1/||h_1||_{\mathfrak{H}}))h_1,$$

and

$$\|DI_n(h)\|_{\mathfrak{H}}^2 = n^2||h_1||_{\mathfrak{H}}^{2n} H_{n-1}(W(h_1/||h_1||_{\mathfrak{H}}))^2.$$

Therefore

$$E(\|DI_n(h)\|_{\mathfrak{H}}^2) = n^2||h_1||_{\mathfrak{H}}^{2n}(n-1)! = nn!||h||_{\mathfrak{H}^{\otimes n}}^2$$

■

Let $u$ be an element of $L^2(\Omega, \mathfrak{H})$ and assume there is an element $\delta(u) \in L^2(\Omega)$ such that

$$E(\langle DF, u\rangle_{\mathfrak{H}}) = E(F\delta(u))$$

for any $F \in \mathbb{D}^{1,2}$, then we say that $u$ is in the domain of $\delta$ and that $\delta$ is the adjoint operator of $D$.

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## Proposition

*Let $h$ be an element of $\mathfrak{H}$,*

$$\delta(h) = W(h)$$

Proof. Without loss of generality we can assume that $||h||_{\mathfrak{H}} = 1$ and that $F = f(W(h), W(h_2), ..., W(h_n))$ with $h_i$ orthogonal to $h$. Then

$$
\begin{aligned}
& E(\langle DF, h\rangle_{\mathfrak{H}}) \\
= \quad & E(\partial_1 f) = E(\int_{\mathbb{R}} \partial_1 f(x_1, W(h_2), ..., W(h_n)) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} dx_1)
\end{aligned}
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= \ & E(FW(h)).
\end{aligned}
$$

■

## Proposition

*If*

$$u = \sum_{i=1}^{n} F_j h_j$$

*where* $F_j$ *are smooth random variables and* $h_j$ *are elements of* $\mathfrak{H}$ *then*

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_{\mathfrak{H}}$$

# Basic Malliavin Calculus

## Divergence operator. Useful formulas

Proof. Let $T \in \mathcal{S}$,

$$E(T\delta(u)) = \sum_{j=1}^{n} E(TF_j W(h_j)) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}})$$

**Proof**. Let $T \in \mathcal{S}$,

$$\begin{aligned} E(T\delta(u)) &= \sum_{j=1}^{n} E(TF_j W(h_j)) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}}) \\ &= \sum_{j=1}^{n} E(\langle D(TF_j), h_j\rangle_{\mathfrak{H}}) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}}) \end{aligned}$$

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# Basic Malliavin Calculus

## Divergence operator. Useful formulas

**Proof**. Let $T \in \mathcal{S}$,

$$
\begin{aligned}
E(T\delta(u)) &= \sum_{j=1}^{n} E(TF_j W(h_j)) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}}) \\
&= \sum_{j=1}^{n} E(\langle D(TF_j), h_j\rangle_{\mathfrak{H}}) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}}) \\
&= \sum_{j=1}^{n} E(\langle D(TF_j) - TDF_j, h_j\rangle_{\mathfrak{H}}) \\
&= \sum_{j=1}^{n} E(F_j\langle DT, h_j\rangle_{\mathfrak{H}}) = E(\langle DT, \sum_{j=1}^{n} F_j h_j\rangle_{\mathfrak{H}})
\end{aligned}
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&= E(\langle DT, u\rangle_{\mathfrak{H}})
\end{aligned}
$$

■

## Proposition

*If*

$$u = H_{n-1}(W(h))h$$

*where* $h \in \mathfrak{H}$, $\|h\|_{\mathfrak{H}} = 1$ *then*

$$\delta(u) = H_n(W(h)).$$

**Proof.**

$$\delta(u) = H_{n-1}(W(h))W(h) - \langle DH_{n-1}(W(h)), h \rangle_{\mathfrak{H}}$$

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**Corollary**

*Let $F \in \mathcal{H}_n$ then*

$$\delta DF = nF.$$

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**Proof**. It is sufficient to consider $F$ of the form $F = H_n(W(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1$. Then

$$\delta DF = \delta(nH_{n-1}(W(h))h) = nH_n(W(h)) = nF.$$

■

## Lemma

*Consider two random variables $F = I_n(f)$, $G = I_m(g)$, where $n, m \geq 1$. Then*

$$E(\langle DF, DG\rangle_{\mathfrak{H}}^2) = \sum_{r=1}^{n\wedge m} \frac{(n!m!)^2}{((n-r)!(m-r)!(r-1)!)^2} \left\| f\widetilde{\otimes}_r g\right\|^2_{\mathfrak{H}^{\odot(n+m-2r)}}.$$

Proof. It is sufficient to consider $f = \widetilde{e^{\otimes a}}$ and $g = \widetilde{e^{\otimes b}}$, then

$$\begin{aligned} DI_n(f) &= \sum_{j=1}^{\infty} a_j I_{n-1}(\widetilde{e^{\otimes a'(j)}}) e_j, \\ DI_m(g) &= \sum_{k=1}^{\infty} b_k I_{m-1}(\widetilde{e^{\otimes b'(k)}}) e_k, \end{aligned}$$

and

$$\begin{aligned} &\langle DI_n(f), DI_m(g)\rangle_{\mathfrak{H}} \\ = &\sum_{k=1}^{\infty} a_k b_k I_{n-1}(\widetilde{e^{\otimes a'(j)}}) I_{m-1}(\widetilde{e^{\otimes b'(k)}}) \end{aligned}$$

**Proof.** It is sufficient to consider $f = \widetilde{e^{\otimes a}}$ and $g = \widetilde{e^{\otimes b}}$, then

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= & \sum_{k=1}^{\infty} a_k b_k \sum_{r=0}^{m \wedge n-1} r! \begin{pmatrix} n-1 \\ r \end{pmatrix} \begin{pmatrix} m-1 \\ r \end{pmatrix} I_{m+n-2-2r}(\widetilde{e^{\otimes a'(k)}} \widetilde{\otimes}_r \widetilde{e^{\otimes b'(k)}}).
\end{aligned}
$$

# Basic Malliavin Calculus

## Useful formulas

Proof (cont.). Hence,

$$\langle DI_n(f), DI_m(g)\rangle_{\mathfrak{H}}$$

$$= \sum_{r=0}^{m\wedge n-1} r! \begin{pmatrix} n-1 \\ r \end{pmatrix} \begin{pmatrix} m-1 \\ r \end{pmatrix} I_{m+n-2-2r}\left(\sum_{k=1}^{\infty} a_k b_k \widetilde{e^{\otimes a'(k)}} \widetilde{\otimes}_r \widetilde{e^{\otimes b'(k)}}\right)$$

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= &\; nm \sum_{r=0}^{m\wedge n-1} r! \binom{n-1}{r}\binom{m-1}{r} I_{m+n-2-2r}\big(\widetilde{e^{\otimes a}} \widetilde{\otimes}_{r+1} \widetilde{e^{\otimes b}}\big).
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\end{aligned}
$$

Finally

$$
\begin{aligned}
&E\left(\langle DI_n(f), DI_m(g)\rangle_{\mathfrak{H}}\right)^2 \\
=\; & (nm)^2 \sum_{r=0}^{m\wedge n-1} (r!)^2 \binom{n-1}{r}^2\binom{m-1}{r}^2 \left\| \widetilde{e^{\otimes a}} \widetilde{\otimes}_{r+1} \widetilde{e^{\otimes b}} \right\|^2_{\mathfrak{H}^{\odot(n+m-2-2r)}}
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■

# CLT. Random variables in a fixed chaos

### Theorem

*Fix* $n \geq 2$*. Consider a sequence* $\{F_k = I_n(f_k), k \geq 1\}$ *such that*

$$E(F_k^2) \underset{k\to\infty}{\to} \sigma^2 \tag{1}$$

# CLT. Random variables in a fixed chaos

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(iii) $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(n-r)}} \underset{k\to\infty}{\to} 0$, *for all* $1 \leq r \leq n-1$ .

# CLT. Random variables in a fixed chaos

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*(iii)* $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(n-r)}} \underset{k\to\infty}{\to} 0$, *for all* $1 \leq r \leq n-1$ .

*(iv)* $\|DF_k\|_{\mathfrak{H}}^2 \overset{L^2(\Omega)}{\underset{k\to\infty}{\to}} n\sigma^2$.

# CLT. Random variables in a fixed chaos

Proof. To simplify we take $\sigma^2 = 1$. We shall prove the following implications

$$(iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

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$(iv) \Rightarrow (i)$. First the sequence $(F_k)$ is tight since it is bounded in $L^2(\Omega)$ by condition (1).

# CLT. Random variables in a fixed chaos

Proof. To simplify we take $\sigma^2 = 1$. We shall prove the following implications

$$(iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

$(iv) \Rightarrow (i)$. First the sequence $(F_k)$ is tight since it is bounded in $L^2(\Omega)$ by condition (1).Second, assume that $F_k$ converges to $G$. Again by (1), $G \in L^2(\Omega)$. Then its characteristic function $\varphi(t) = E(e^{itG})$ is differentiable and $\varphi'(t) = iE(Ge^{itG})$. For every $k \geq 1$, define $\varphi_k(t) = E(e^{itF_k})$, then $\varphi_k'(t) = iE(F_k e^{itF_k})$. Clearly, $F_k e^{itF_k}$ converges in law to $Ge^{itG}$ and the boundedness in $L^2(\Omega)$, implies convergence of the first order moments.

# CLT. Random variables in a fixed chaos

Proof, $(iv) \Rightarrow (i)(cont.)$ . Then, $\varphi_k'(t) \to \varphi'(t)$.

# CLT. Random variables in a fixed chaos

Proof, $(iv) \Rightarrow (i)(cont.)$ . Then, $\varphi_k'(t) \to \varphi'(t)$. Moreover,

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# CLT. Random variables in a fixed chaos

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$$\begin{aligned} \varphi_k'(t) &= iE(F_k e^{itF_k}) = \frac{i}{n} E(\delta D(F_k) e^{itF_k}) \\ &= -\frac{t}{n} E(e^{itF_k} \langle DF_k, DF_k \rangle_{\mathfrak{H}}) \end{aligned}$$

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\end{aligned}$$

in fact, by (iv)

$$\left| E(e^{itF_k} \|DF_k\|_{\mathfrak{H}}^2) - n\varphi(t) \right| \leq E\left( \left| \|DF_k\|_{\mathfrak{H}}^2 - n \right| \right) + n \left| E(e^{itF_k}) - \varphi(t) \right| .$$

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$$

This implies that $\varphi(t)$ satisfies the differential equation

$$
\varphi'(t) = -t\varphi(t), \varphi(0) = 1.
$$

# CLT. Random variables in a fixed chaos

Proof. $(i) \Rightarrow (ii)$. It is well known that, for any $1 < p < q < \infty$ the norms $\|\cdot\|_p$, $\|\cdot\|_q$ are equivalent in any Wiener chaos $\mathcal{H}_n$, then convergence in law and convergence of the second order moments implies convergence of the moments of any order.

# CLT. Random variables in a fixed chaos

Proof. $(ii) \Rightarrow (iii)$. By using the product formula

$$
\begin{aligned}
I_n(f_k)^2 &= \sum_{r=0}^{n} r! \binom{n}{r}^2 I_{2(n-r)}(f_k \widetilde{\otimes}_r f_k) \\
&= n! ||f_k||^2 + I_{2n}\left(f_k \widetilde{\otimes} f_k\right) + \sum_{r=1}^{n-1} r! \binom{n}{r}^2 I_{2(n-r)}(f_k \widetilde{\otimes}_r f_k).
\end{aligned}
$$

Then

# CLT. Random variables in a fixed chaos

Proof. (*ii*) $\Rightarrow$ (*iii*).By using the product formula

$$I_n(f_k)^2 = \sum_{r=0}^{n} r! \binom{n}{r}^2 I_{2(n-r)}(f_k \widetilde{\otimes}_r f_k)$$

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Then

$$E(I_n(f_k)^4) = (n!)^2 \, ||f_k||^4$$
$$+(2n)! \left\| f_k \widetilde{\otimes} f_k \right\|^2_{\mathfrak{H}^{\otimes 2n}} + \sum_{r=1}^{n-1} (r!)^2 \binom{n}{r}^4 (2(n-r))! \left\| f_k \widetilde{\otimes}_r f_k \right\|^2_{\mathfrak{H}^{\otimes 2(n-r)}},$$

also we have that

# CLT. Random variables in a fixed chaos

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also we have that

$$||f_k \widetilde{\otimes} f_k||^2_{\mathfrak{H}^{\otimes 2n}} = f_k \otimes f_k \otimes_{2n} f_k \widetilde{\otimes} f_k.$$

# CLT. Random variables in a fixed chaos

Proof. $(ii) \Rightarrow (iii)(cont.)$. Therefore, $\left\| f_k \widetilde{\otimes} f_k \right\|^2_{\mathfrak{H}^{\otimes 2n}}$ is the sum of $\frac{(2n)!}{(n!)^2}$ terms of the form $\frac{(n!)^2}{(2n)!} \| f_k \otimes_a f_k \|^2$ with $a = 0, 1, .., n$. And for $a = 0, n$

$$\| f_k \otimes_a f_k \|^2_{\mathfrak{H}^{\otimes 2(n-a)}} = \| f_k \|^4.$$

Consequently

$$E(I_n(f_k)^4) = 3\,(n!)^2\,\| f_k \|^4 + R_k,$$

and, by the hypothesis (ii) $R_k \to 0$, equivalently $\| f_k \otimes_r f_k \|^2_{\mathfrak{H}^{\otimes 2(n-r)}} \to 0, 1 \leq r \leq n-1$.

# CLT. Random variables in a fixed chaos

Proof. (*iii*) $\Rightarrow$ (*iv*). We know that $E(\|DI_n(h)\|^2_{\mathfrak{H}}) = nn!\,\|h\|^2_{\mathfrak{H}}$, then

$$E((\|DF_k\|^2_{\mathfrak{H}} - n)^2) = E(\|DF_k\|^4_{\mathfrak{H}}) - 2n^2 n!\,\|f_k\|^2_{\mathfrak{H}^{\otimes n}} + n^2,$$

therefore it suffices to prove that (*iii*) implies that $E(\|DF_k\|^4_{\mathfrak{H}}) \to n^2$. But, by the previous Lemma we have that

$$E(\|DF_k\|^4_{\mathfrak{H}}) = \sum_{r=1}^{n-1} \frac{(n!)^4}{\left(((n-r)!)^2 (r-1)!\right)^2} \left\|f_k \widetilde{\otimes}_r f_k\right\|_{\mathfrak{H}^{\odot 2(n-r)}} + n^2 (n!)^2 \|f_k\|^4_{\mathfrak{H}^{\otimes n}},$$

so, by (*iii*), $E(\|DF_k\|^4_{\mathfrak{H}}) \to n^2$. ■

# CLT. Random variables in a fixed chaos

Example

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\mathfrak{H} = L^2([0,1], dx)$. Then

$$F_k := \sqrt{k}\left(\frac{1}{k}\int_0^1 B_t^2 t^{1/k-2} dt - 1\right) \underset{k\to\infty}{\to} N(0,2).$$

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It is easy to see that

$$\begin{aligned} F_k &= \frac{\sqrt{k}}{k-1}\int_0^1\int_0^1 \left((s\vee t)^{1/k-1} - 1\right) dB_s dB_t \\ &= I_2(f_k), \end{aligned}$$

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where

$$f_k = \frac{\sqrt{k}}{k-1}\left((s\vee t)^{1/k-1} - 1\right) \in \mathfrak{H}^{\odot 2}$$

# CLT. Random variables in a fixed chaos

Example (cont.) We have that

$$\begin{aligned} E(F_k^2) &= 2||f_k||^2_{\mathfrak{H}^{\otimes 2}} = \frac{2k}{(k-1)^2}\int_0^1\int_0^1 \left((s\vee t)^{1/k-1}-1\right) dsdt \\ &= \frac{4k}{(k-1)^2}\left(\frac{k}{2}-\frac{2k}{k+1}+\frac{1}{2}\right) \underset{k\to\infty}{\longrightarrow} 2. \end{aligned}$$

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Then it is enough to see that

$$||f_k \otimes_1 f_k||^2_{\mathfrak{H}^{\otimes 2}} \underset{k\to\infty}{\to} 0.$$

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But

$$\begin{aligned} & ||f_k \otimes_1 f_k||^2_{\mathfrak{H}^{\otimes 2}} \\ = \; & \frac{k^2}{(k-1)^4}\int_0^1\int_0^1\left(\int_0^1 \left((s\vee t)^{1/k-1}-1\right)\left((s\vee u)^{1/k-1}-1\right) ds\right)^2 dtdu \end{aligned}$$

and $\int_0^1\int_0^1\left(\int_0^1 \left((s\vee t)^{1/k-1}-1\right)\left((s\vee u)^{1/k-1}-1\right) ds\right)^2 dtdu = O(k)$.

# CLT. Random variables in a fixed chaos

## Example

Consider a sequence of stationary, normalized, centered Gaussian random variables $(X_i)_{i\geq 1}$. We want to study the asymptotic behavior of the sequence

$$F_k := \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) \; ,$$

$m \geq 2$. We can take $\mathcal{H}_1 = span\{X_i, i \geq 1\}$, and $\mathfrak{H} \equiv \mathcal{H}_1$. The inner product on $\mathfrak{H}$ is then induced by the covariance function $\rho(k) = \mathrm{cov}(X_1, X_{1+k})$ of the sequence $(X_i)_{i\geq 1}$ (note that $\rho(0) = 1$).

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$$F_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) = I_m \left( \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i^{\otimes m} \right) .$$

# CLT. Random variables in a fixed chaos

## Example

Consider a sequence of stationary, normalized, centered Gaussian random variables $(X_i)_{i\geq 1}$. We want to study the asymptotic behavior of the sequence

$$F_k := \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) \, ,$$

$m \geq 2$. We can take $\mathcal{H}_1 = span\{X_i, i \geq 1\}$, and $\mathfrak{H} \equiv \mathcal{H}_1$. The inner product on $\mathfrak{H}$ is then induced by the covariance function $\rho(k) = \text{cov}(X_1, X_{1+k})$ of the sequence $(X_i)_{i\geq 1}$ (note that $\rho(0) = 1$).We obtain the following representation

$$F_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) = I_m \left( \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i^{\otimes m} \right) .$$

Set

$$h_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i^{\otimes m}.$$

# CLT. Random variables in a fixed chaos

Example (cont.) Assume that

$$\sum_{j=1}^{\infty} |\rho(j)|^m < \infty. \tag{2}$$

It holds that

$$m!||h_k||^2_{\mathfrak{H}^{\otimes m}} = \frac{m!}{k} \sum_{i,j=1}^{k} (E(X_i X_j))^m = \frac{m!}{k} \sum_{i,j=1}^{k} \rho^m(i-j)$$

# CLT. Random variables in a fixed chaos

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$$= \quad m! \left( 1 + 2 \sum_{j=1}^{k-1} \rho^m(j) \left( 1 - \frac{j}{k} \right) \right) \to m! \left( 1 + 2 \sum_{j=1}^{\infty} \rho^m(j) \right) =: \sigma^2.$$

Note the identity

$$h_k \otimes_r h_k = \frac{1}{k} \sum_{i,j=1}^{k} \rho^r(i-j) X_i^{\otimes(m-r)} \otimes X_j^{\otimes(m-r)},$$

# CLT. Random variables in a fixed chaos

## Example (cont.)

This implies

$$
\begin{aligned}
&\|h_k \otimes_r h_k\|^2_{\mathfrak{H}^{\otimes 2(m-r)}} \\
= \quad & \frac{1}{k^2} \sum_{i,j,i',j'=1}^{k} \rho^r(i-j)\rho^r(i'-j')\rho^{m-r}(i-i')\rho^{m-r}(j-j') \\
= \quad & \frac{1}{k} \sum_{i,j,i'=0}^{k-1} \rho^r(i)\rho^r(j-i')\rho^{m-r}(j)\rho^{m-r}(i-i')\left(1-\frac{i \vee j \vee i'}{k}\right) \\
\leq \quad & \frac{1}{k} \sum_{i,j,i'=0}^{k-1} \rho(i)\rho(j-i')\rho(j)\rho(i-i') \\
= \quad & \frac{1}{k} \sum_{i=0}^{k-1} \left( \sum_{j=0}^{k-1} \rho(j)\rho(i-j) \right)^2 \leq 2\varepsilon \left( \sum_{j=0}^{\infty} \rho(j)^2 \right)^2,
\end{aligned}
$$

for any $\varepsilon > 0$. So the last term converges to 0 under assumption (2) for $1 \leq r \leq m-1$, and we deduce that $F_k \xrightarrow[k\to\infty]{\mathcal{L}} N(0, \sigma^2)$.

# CLT. Random vectors with components in fixed chaos

For $d \geq 2$, fix $d$ natural numbers, $1 \leq n_1 \leq ... \leq n_d$. Consider a sequence of random vectors

$$F_k = (F_k^1, .., F_k^d) = (I_{n_1}(f_k^1), ..., I_{n_d}(f_k^d)), \quad (3)$$

where $f_k^i \in \mathfrak{H}^{\odot n_i}$. We have a multidimensional version of the previous theorem,

# CLT. Random vectors with components in fixed chaos

## Theorem

*Let* $(F_k)_{k\geq 1}$ *be a sequence of random vectors of the form (3) such that, for every* $1 \leq i, j \leq d$

$$\lim E(F_k^i F_k^j) = \delta_{ij}, \tag{4}$$

# CLT. Random vectors with components in fixed chaos

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# CLT. Random vectors with components in fixed chaos

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*(i) For every* $i = 1, ..., d$, $F_k^i \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} N(0,1)$.

*(ii) For every* $i = 1, ..., d$, $E\left(\left(F_k^i\right)^4\right) \underset{k\to\infty}{\to} 3$.

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*(i) For every* $i = 1, ..., d,\ F_k^i \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} N(0,1).$

*(ii) For every* $i = 1, ..., d,\ E\left(\left(F_k^i\right)^4\right) \underset{k\to\infty}{\to} 3.$

*(iii)* $\left\|f_k^i \otimes_r f_k^i\right\|_{\mathfrak{H}^{\otimes 2(n_i - r)}} \underset{k\to\infty}{\to} 0,$ *for all* $1 \leq r \leq n_i - 1, 1 \leq i \leq d$ .

# CLT. Random vectors with components in fixed chaos

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*(iv) For every* $i = 1, ..., d, \left\|DF_k^i\right\|_{\mathfrak{H}}^2 \underset{k\to\infty}{\overset{L^2(\Omega)}{\to}} n_i$.

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*(iv) For every* $i = 1, ..., d, \left\|DF_k^i\right\|_{\mathfrak{H}}^2 \underset{k\to\infty}{\overset{L^2(\Omega)}{\to}} n_i$.

*(v)* $F_k \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} N_d(0, I_d)$

# CLT. Random vectors with components in fixed chaos

Proof.$(iv) \Rightarrow (v)$. Assume then that $F_k$ converges to $G$ and that $\varphi(t) = E(e^{i\langle t,G\rangle})$ . Set $\varphi_k(t) = E(e^{i\langle t,F_k\rangle})$, then $\partial_j\varphi_k(t) = iE(F_k^j e^{i\langle t,F_k\rangle}) \to \partial_j\varphi(t)$.

# CLT. Random vectors with components in fixed chaos

Proof.$(iv) \Rightarrow (v)$. Assume then that $F_k$ converges to $G$ and that $\varphi(t) = E(e^{i\langle t, G\rangle})$ . Set $\varphi_k(t) = E(e^{i\langle t, F_k\rangle})$, then $\partial_j \varphi_k(t) = iE(F_k^j e^{i\langle t, F_k\rangle}) \to \partial_j \varphi(t)$. Moreover, by (iv),

$$
\begin{aligned}
\partial_j \varphi_k(t) &= iE(F_k^j e^{i\langle t, F_k\rangle}) = \frac{i}{n_j} E(\delta D(F_k^j) e^{i\langle t, F_k\rangle}) \\
&= -\frac{1}{n_j} \sum_{l=1}^{d} t_l E(e^{i\langle t, F_k\rangle} \langle DF_k^j, DF_k^l\rangle_{\mathfrak{H}}) \\
&= -\frac{t_j}{n_j} E(e^{i\langle t, F_k\rangle} \left\| DF_k^j \right\|_{\mathfrak{H}}^2) \to -t_j \varphi(t),
\end{aligned}
$$

in fact, for $j \neq l$, by using the Lemma and condition 4 it is easy to see that

$$E(\langle DF_k^j, DF_k^l\rangle_{\mathfrak{H}}^2) \to 0.$$

# CLT. Random vectors with components in fixed chaos

**Proof.** $(iv) \Rightarrow (v)$. Assume then that $F_k$ converges to $G$ and that $\varphi(t) = E(e^{i\langle t,G\rangle})$ . Set $\varphi_k(t) = E(e^{i\langle t,F_k\rangle})$, then $\partial_j \varphi_k(t) = iE(F_k^j e^{i\langle t,F_k\rangle}) \to \partial_j \varphi(t)$. Moreover, by (iv),

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\end{aligned}$$

in fact, for $j \neq l$, by using the Lemma and condition 4 it is easy to see that

$$E(\langle DF_k^j, DF_k^l\rangle_{\mathfrak{H}}^2) \to 0.$$

This implies that $\varphi(t)$ satisfies the partial differential equation

$$\begin{aligned}
\partial_j \varphi(t) &= -t_j \varphi(t), \ j = 1, ..., d. \\
\varphi(0) &= 1.
\end{aligned}$$

■

# CLT. General random vectors

Finally, we can consider a $d$-dimensional random vector $F_k = (Y_k^1, \ldots, Y_k^d)^T$ which has a chaos representation

$$F_k^i = \sum_{m=1}^{\infty} I_m(f_{m,k}^i) , \qquad i = 1, \ldots, d ,$$

with $f_{m,k}^i \in \mathfrak{H}^{\odot m}$.

# CLT. General random vectors

## Theorem

*Suppose that the following conditions hold:*

(i) *For any* $i = 1, \ldots, d$ *we have* $\sum_{m=1}^{\infty} \sup_k m! ||f^i_{m,k}||^2_{\mathfrak{H}^{\otimes m}} < \infty$.

(ii) *For any* $m \geq 1$, $i, j = 1, \ldots, d$ *we have constants* $\Sigma^m_{ij}$ *such that*

$$\lim_{k \to \infty} E[I_m(f^i_{m,k}) I_m(f^j_{m,k})] = \lim_{k \to \infty} \left\langle f^i_{m,k}, f^j_{m,k} \right\rangle_{\mathfrak{H}^{\odot m}} = \Sigma^m_{ij} \,,$$

*and the matrix* $\Sigma^m = (\Sigma^m_{ij})_{1 \leq i,j \leq d}$ *is positive definite for all* $m$.

(iii) $\sum_{m=1}^{\infty} \Sigma^m = \Sigma \in \mathbb{R}^{d \times d}$.

(iv) *For any* $m \geq 1$, $i = 1, \ldots, d$ *and* $p = 1, \ldots, m-1$

$$\lim_{k \to \infty} ||f^i_{m,k} \otimes_p f^i_{m,k}||^2_{\mathfrak{H}^{\otimes 2(m-p)}} = 0.$$

# CLT. General random vectors

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$$\lim_{k\to\infty} E[I_m(f^i_{m,k})I_m(f^j_{m,k})] = \lim_{k\to\infty} \left\langle f^i_{m,k}, f^j_{m,k} \right\rangle_{\mathfrak{H}^{\odot m}} = \Sigma^m_{ij} ,$$

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$$\lim_{k\to\infty} ||f^i_{m,k} \otimes_p f^i_{m,k}||^2_{\mathfrak{H}^{\otimes 2(m-p)}} = 0.$$

*Then we have* $F_k \xrightarrow[k\to\infty]{\mathcal{L}} N_d(0, \Sigma)$.

# CLT. General random vectors

Proof. Fix $v \in \mathbb{R}^d$. By the theorem for the unidimensional case and condition (ii) and (iv) we have that $I_m(v^T f_{m,k})$ converges to a $N(0, v^T \Sigma^m v)$ as $k$ goes to infinity.

# CLT. General random vectors

Proof. Fix $\upsilon \in \mathbb{R}^d$. By the theorem for the unidimensional case and condition (ii) and (iv) we have that $I_m(\upsilon^T f_{m,k})$ converges to a $N(0, \upsilon^T \Sigma^m \upsilon)$ as $k$ goes to infinity. Then if use the theorem for the multidimensional case

$$\Big(I_1(\upsilon^T f_{1,k}), ..., I_m(\upsilon^T f_{m,k})\Big) \xrightarrow[k\to\infty]{\mathcal{L}} (\xi_1, ..., \xi_m), \tag{5}$$

where, for $i \geq 1$, $\xi_i$ are independent $N(0, \upsilon^T \Sigma^i \upsilon)$.

# CLT. General random vectors

Proof. Fix $\upsilon \in \mathbb{R}^d$. By the theorem for the unidimensional case and condition (ii) and (iv) we have that $I_m(\upsilon^T f_{m,k})$ converges to a $N(0, \upsilon^T \Sigma^m \upsilon)$ as $k$ goes to infinity. Then if use the theorem for the multidimensional case

$$\Big(I_1(\upsilon^T f_{1,k}), \ldots, I_m(\upsilon^T f_{m,k})\Big) \xrightarrow[k\to\infty]{\mathcal{L}} (\xi_1, \ldots, \xi_m), \tag{5}$$

where, for $i \geq 1$, $\xi_i$ are independent $N(0, \upsilon^T \Sigma^i \upsilon)$. Define for every $N \geq 1$,

$$\begin{aligned} F_k^N &= \sum_{m=1}^{N} I_m(f_{m,k}) \;, \\ \xi^N &= \sum_{m=1}^{N} \xi_m \end{aligned}$$

# CLT. General random vectors

Proof (cont.). Set also $\xi = \sum_{m=1}^{\infty} \xi_m$. Let $f \in C^1$ bounded and with bounded derivative, then

$$\begin{aligned}
\left|E(f(\upsilon^T F_k) - f(\xi))\right| &\leq \left|E(f(\upsilon^T F_k) - f(\upsilon^T F_k^N))\right| \\
&\quad + \left|E(f(\upsilon^T F_k^N) - f(\xi^N))\right| + \left|E(f(\xi) - f(\xi^N))\right| \\
&\leq C|\upsilon| \left( \sum_{m=N+1}^{\infty} E(I_m(f_{m,k})^2) \right)^{1/2} \\
&\quad + \left|E(f(\upsilon^T F_k^N) - f(\xi^N))\right| + \left|E(f(\xi) - f(\xi^N))\right|.
\end{aligned}$$

So, by conditions (i), 5 and (iii), if we take the supremum in $k$ and then the limit in $N$ we obtain the result. ■

# CTL for random processes. The power variation

Let $(G_t)_{t\geq 0}$ be a Gaussian process which has centered and stationary increments. We want to study the asymptotic properties of the process

$$V(G,p)^n_t = \frac{1}{n\tau_n^p}\sum_{i=1}^{[nt]} |\Delta_i^n G|^p ,$$

where $\Delta_i^n G = G_{\frac{i}{n}} - G_{\frac{i-1}{n}}$, $\tau_n^2 = E[|\Delta_i^n G|^2]$ and $p > 0$. Write

$$r_n(j) = \text{Cov}\left(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\right) , \qquad j \geq 0.$$

and assume that

$$|r_n(j)|^2 \leq Cj^{-1-\varepsilon},\ j \geq 0, \text{for some } \varepsilon > 0 \tag{6}$$

and

$$\lim_{n\to\infty} r_n(j) = \rho(j),$$

Set $H(x) = |x|^p - \mu_p$, where $\mu_p = E(|N(0,1)|^p)$, then $H(x) = \sum_{j=2}^{\infty} a_j H_j(x)$, with $a_2 > 0$ and we have the following theorem:

# CTL for random processes. The power variation

### Theorem

$$\left(G_t, \sqrt{n}(V(G,p)_t^n - t\mu_p)\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \left(G_t, \sigma W_t\right) ,$$

*where $W$ is a Brownian motion that is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, independent of $G$ and $\sigma^2$ is given by*

$$\sigma^2 = \sum_{m=2}^{\infty} \sigma_m^2, \quad \sigma_m^2 = m! a_m^2 \lambda_m^2 , \quad \lambda_m^2 = 1 + 2\sum_{i=1}^{\infty} \rho^m(i). \quad (7)$$

# CTL for random processes. The power variation

Proof. First we have to show the convergence of the f.d.d. Let $(a_k, b_k]$ pairwise disjoint intervals in $[0, T]$. Define

$$G_n^k = \tau_n \sum_{i=[na_k]+1}^{[nb_k]} \frac{\Delta_i^n G}{\tau_n},$$

$$Y_n^k = \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big),$$

it suffices to prove that

$$\Big(G_n^k, Y_n^k\Big)_{1\le k\le d} \xrightarrow{\mathcal{L}} \Big(G_{b_k} - G_{a_k}, \sigma(W_{b_k} - W_{a_k})\Big)_{1\le k\le d},$$

where $\sigma$ is given by (7) and $W$ is independent of $G$.

# CTL for random processes. The power variation

Proof (cont.). Let $\mathcal{H}_1$ the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nT]}$. Notice that $\mathcal{H}_1$ is a separable Hilbert space with the scalar product induced by the covariance function of the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nT]}$. Then we can take $\mathfrak{H} = \mathcal{H}_1$ and try to apply the general CLT to this case.

# CTL for random processes. The power variation

Proof (cont.).

We have

$$Y_n^k = \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big)$$

# CTL for random processes. The power variation

Proof (cont.).

We have

$$
\begin{aligned}
Y_n^k &= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\left(\frac{\Delta_i^n G}{\tau_n}\right) \\
&= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \sum_{m=2}^{\infty} a_m H_m \left(\frac{\Delta_i^n G}{\tau_n}\right)
\end{aligned}
$$

# CTL for random processes. The power variation

Proof (cont.).

We have

$$
\begin{aligned}
Y_n^k &= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big) \\
&= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \sum_{m=2}^{\infty} a_m H_m \left(\frac{\Delta_i^n G}{\tau_n}\right) \\
&= \sum_{m=2}^{\infty} I_m \left( \frac{a_m}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left(\frac{\Delta_i^n G}{\tau_n}\right)^{\otimes m} \right),
\end{aligned}
$$

# CTL for random processes. The power variation

Proof (cont.).

We have

$$
\begin{aligned}
Y_n^k &= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big) \\
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&= \sum_{m=2}^{\infty} I_m \left( \frac{a_m}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left(\frac{\Delta_i^n G}{\tau_n}\right)^{\otimes m} \right),
\end{aligned}
$$

and

$$
G_n^k = \tau_n \sum_{i=[na_k]+1}^{[nb_k]} \frac{\Delta_i^n G}{\tau_n} = I_1 \left( \tau_n \sum_{i=[na_k]+1}^{[nb_k]} \frac{\Delta_i^n G}{\tau_n} \right),
$$

# CTL for random processes. The power variation

Proof (cont.).
The components $(Y_n^k)$ and $(G_n^k)$ are orthogonals and it is clear that $(G_n^k) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} (G_{b_k} - G_{a_k})$.

# CTL for random processes. The power variation

Proof (cont.).
The components $(Y_n^k)$ and $(G_n^k)$ are orthogonals and it is clear that $(G_n^k) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} (G_{b_k} - G_{a_k})$.So we have just to prove that $(Y_n^k)$ $\underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} N_d(0, \sigma^2 I_d)$.
Then we can apply the previous theorem with

$$f_{m,n}^k = \frac{a_m}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left( \frac{\Delta_i^n G}{\tau_n} \right)^{\otimes m} \in \mathfrak{H}^{\odot m}.$$

# CTL for random processes. The power variation

Proof (i). Take, by simplicity, $k = 1, a_1 = 0, b_1 = 1$

$$||f^1_{m,n}||^2_{\mathfrak{H}^{\otimes m}} = \frac{a_m^2 m!}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} r_n^m(|i-j|)$$

$$= a_m^2 m!(1 + 2\sum_{i=1}^{n-1}(1 - \frac{i}{n}) r_n^m(i)) \quad (8)$$

$$\underset{n\to\infty}{\longrightarrow} a_m^2 m!(1 + 2\sum_{i=1}^{\infty} \rho^m(i)) = \sigma_m^2. \quad (9)$$

# CTL for random processes. The power variation

Proof (ii) and (iii). Take again by simplicity and w.l.o.g $k = 1, a_1 = 0, b_1 = a_1 = 1$ and $b_2 = 2$

$$
\begin{aligned}
\left| \left\langle f^1_{m,n}, f^2_{m,n} \right\rangle_{\mathfrak{H}^{\odot m}} \right| &= \left| \frac{a_m^2 m!}{n} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} r_n^m(j-i) \right| \\
&= \left| \frac{a_m^2 m!}{n} \left( \sum_{j=1}^{n} j r_n^m(j) + \sum_{j=1}^{n-1} j r_n^m(2n-j) \right) \right| \\
&\leq \frac{a_m^2 m!}{n} \left( \sum_{j=1}^{n} j r^m(j) + \sum_{j=1}^{n-1} j r^m(2n-j) \right) \\
&\underset{n\to\infty}{\rightarrow} 0 \text{ (since } n r^m(n) \underset{n\to\infty}{\rightarrow} 0).
\end{aligned}
$$

# CTL for random processes. The power variation

Proof (iv). Fix $1 \leq p \leq m-1$.

$$f^1_{m,n} \otimes_p f^1_{m,n} = \frac{1}{n} \sum_{i,j=1}^{n} r_n^p(|i-j|) \left( \frac{\Delta_i^n G}{\tau_n} \right)^{\otimes(m-r)} \otimes_p \left( \frac{\Delta_j^n G}{\tau_n} \right)^{\otimes(m-r)},$$

and we have, like in the second example, that

$$\left\| f^1_{m,n} \otimes_p f^1_{m,n} \right\|^2 \leq 2\varepsilon \left( \sum_{l=1}^{\infty} \rho(l)^2 \right)^2.$$

# CTL for random processes. The power variation

Proof (tightness). The second step of the proof is to check the tightness condition of $(G_t, Z_t^{(n)})$ where

$$Z_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big) \; .$$

Set

$$Z_t^{n,N} := \sum_{m=2}^{N} I_m\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m}\Big).$$

# CTL for random processes. The power variation

**Proof (tightness) (cont.)**
then we have, for $s < t$,

$$
\begin{aligned}
& E(|Z_t^{n,N} - Z_s^{n,N}|^2) \\
= \; & E(\sum_{m=2}^{N} I_m \Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-[ns]} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m}\Big)^2)
\end{aligned}
$$

# CTL for random processes. The power variation

**Proof (tightness) (cont.)**

then we have, for $s < t$,

$$
\begin{aligned}
& E(|Z_t^{n,N} - Z_s^{n,N}|^2) \\
= \quad & E\Big(\sum_{m=2}^{N} I_m\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-[ns]} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m}\Big)^2\Big) \\
= \quad & \frac{[nt]-[ns]}{n} \sum_{m=2}^{N} \frac{1}{[nt]-[ns]} \sum_{i=1}^{[nt]-[ns]} \sum_{j=1}^{[nt]-[ns]} r_n^m(|i-j|)
\end{aligned}
$$

# CTL for random processes. The power variation

**Proof (tightness) (cont.)**

then we have, for $s < t$,

$$
\begin{aligned}
& E(|Z_t^{n,N} - Z_s^{n,N}|^2) \\
= \; & E\Big(\sum_{m=2}^{N} I_m\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-[ns]} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m}\Big)^2\Big) \\
= \; & \frac{[nt]-[ns]}{n} \sum_{m=2}^{N} \frac{1}{[nt]-[ns]} \sum_{i=1}^{[nt]-[ns]} \sum_{j=1}^{[nt]-[ns]} r_n^m(|i-j|) \\
\leq \; & C \frac{[nt]-[ns]}{n}.
\end{aligned}
$$

# CTL for random processes. The power variation

Proof (tightness) (cont.).
By the equivalence of the $L^p$ norms for $1 < p < \infty$ on a fixed sum of Wiener chaos,

$$E(|Z_t^{n,N} - Z_s^{n,N}|^4)^{1/2} \leq C \frac{[nt] - [ns]}{n}$$

# CTL for random processes. The power variation

Proof (tightness) (cont.).
By the equivalence of the $L^p$ norms for $1 < p < \infty$ on a fixed sum of Wiener chaos,

$$E(|Z_t^{n,N} - Z_s^{n,N}|^4)^{1/2} \leq C\frac{[nt]-[ns]}{n}$$

Then by the Cauchy-Schwarz inequality we obtain the approximation

$$P\Big(|Z_t^{n,N} - Z_{t_1}^{n,N}| \geq \lambda, |Z_{t_2}^{n,N} - Z_t^{n,N}| \geq \lambda\Big)$$
$$\leq \quad C\frac{([nt]-[nt_1])([nt_2]-[nt])}{n^2\lambda^4} \leq C\frac{(t_2-t_1)^2}{\lambda^4}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$.

# CTL for random processes. The power variation

Proof (tightness) (cont.).
Moreover we have proved in the first step, by 9 and 6, that

$$\lim_{N\to\infty} \sup_n E[|Z_t^n - Z_t^{n,N}|^2] = 0.$$

Using this we conclude that

$$P\Big(|Z_t^n - Z_{t_1}^n| \geq \lambda, |Z_{t_2}^n - Z_t^n| \geq \lambda\Big) \leq C\frac{(t_2 - t_1)^2}{\lambda^4}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$, from which we deduce the tightness of the sequence $Z_t^n$ by Billingsley's criterium. ■

# CTL for random processes. The bipower variation

If we want to study the asymptotic behaviour of the *bipower variation processes*

$$V(G;p,q)^n_t = \frac{1}{n}\sum_{i=1}^{[nt]} \left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q , \qquad p,q \geq 0 ,$$

we can consider

$$Z^n_t := \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} \left( \left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q - \mu^{(n)}_{p,q} \right) , \qquad p,q \geq 0 ,$$

where $\mu^{(n)}_{p,q} := E\left( \left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q \right)$.

# CTL for random processes. The bipower variation

Then by using the product formula we have

$$Z_t^n = \sum_{m=2}^{\infty} I_m\Big(\frac{1}{n}\sum_{i=1}^{[nt]} f_{m,n}^i\Big),$$

where

$$f_{m,n}^i = \sum_{h=0}^{m} s_{h,m}^{(n)} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes h} \tilde{\otimes} \Big(\frac{\Delta_{i+1}^n G}{\tau_n}\Big)^{\otimes m-h}$$

and

$$s_{h,m}^{(n)} = \sum_{l=0}^{\infty} a_{p,l+h} a_{q,l+m-h} l! \begin{pmatrix} l+h \\ l \end{pmatrix} \begin{pmatrix} l+m-h \\ l \end{pmatrix} r_n^l(1).$$

# CTL for random processes. The bipower variation

Now we can introduce two independent variables $X_i^n(1), X_i^n(2) \sim N(0,1)$ that are given by

$$X_i^n(1) = \frac{\Delta_i^n G}{\tau_n}\ , \qquad X_i^n(2) = a_n \frac{\Delta_i^n G}{\tau_n} + b_n \frac{\Delta_{i+1}^n G}{\tau_n}$$

with $b_n = (1 - r_n^2(1))^{-1/2}$ and $a_n = -(1/r_n^2(1) - 1)^{-1/2}$.

It is clear that $f^i_{m,n}$ can be represented as

$$f^i_{m,n} = \sum_{k_l \in \{1,2\}} c^n_{k_1,\dots,k_m} X^n_i(k_1) \otimes \cdots \otimes X^n_i(k_m) \ ,$$

for some constants $c^n_{k_1,\dots,k_m}$. Note that all summands are orthogonal. We obtain

$$||f^i_{m,n}||^2_{\mathfrak{H}_1^{\otimes m}} = \sum_{k_l \in \{1,2\}} |c^n_{k_1,\dots,k_m}|^2 =: c^n_m.$$

# CTL for random processes. The bipower variation

Also we have that

$$\begin{aligned} & |\langle f^1_{m,n}, f^{1+k}_{m,n} \rangle_{\mathfrak{H}^{\otimes m}}| \\ = & \sum_{h_l \in \{1,2\}, g_l \in \{1,2\}} c^n_{h_1,\ldots,h_m} c^n_{g_1,\ldots,g_m} \prod_{l=1}^{m} \langle X^n_i(h_l), X^n_{i+k}(g_l) \rangle_{\mathcal{H}_1} \\ \leq & \; c^n_m (Cr(k-1))^m. \end{aligned}$$

And by using these results we can prove the central limit theorem for $V(G; p, q)^n_t$.

# CTL for random processes. The multipower variation

A similar extension works for the multipower variation

$$V(G, p_1, \ldots, p_k)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} \left| \frac{\Delta_{i+j-1}^n G}{\tau_n} \right|^{p_j}, \quad p_1, \ldots, p_k \geq 0,$$

and for the **joint** multipower variation:

$$\left( V(G, p_1^1, \ldots, p_k^1)_t^n, ..., V(G, p_1^d, \ldots, p_k^d)_t^n \right).$$

# CTL for random processes. The multipower variation

Define

$$\rho^{(n)}_{p_1,\dots,p_k} = E\left[\left|\frac{\Delta^n_1 G}{\tau n}\right|^{p_1} \cdots \left|\frac{\Delta^n_k G}{\tau n}\right|^{p_k}\right].$$

We have

## Theorem

$$\left(G_t, \sqrt{n}\left(V(p^j_1,\dots,p^j_k)^n_t - \rho^{(n)}_{p^j_1,\dots,p^j_k} t\right)_{1\le j\le d}\right) \to (G_t, \beta^{1/2} W_t),$$

*where $W$ is a $d$-dimensional Brownian, defined in an extension of the original filtered space, independent of $G$, $\beta$ is a $d \times d$-dimensional matrix given by*

$$\beta_{ij} = \lim_{n\to\infty} n\, cov\Big(V_Q(p^i_1,\dots,p^i_k)^n_1, V_Q(p^j_1,\dots,p^j_k)^n_1\Big)\,, \quad 1 \le i,j \le d,$$

*and $(Q_i)_{i\ge 1}$ is stationary centered discrete time Gaussian process with correlation function $\rho(j)$.*

Barndorff-Nielsen, O.E., Corcuera, J.M. and Podolskij, M. (2009a): Power variation for Gaussian processes with stationary increments, *Stoch. Proc. Appl.* **119**, 1845-1865.

Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. and Woerner, J.H.C. (2009b): Bipower variation for Gaussian processes with stationary increments, *J. Appl. Prob.* **46**, 132-150.

Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. (2009c): Multipower variation for Brownian Semistationary processes. Preprint IMUB 412.

Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. (2009d): Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. Preprint IMUB 413.

- Corcuera, J.M. Nualart, D. and Woerner, J.H.C. (2006): Power variation of some integral fractional processes. *Bernoulli* **12**, 713-735.
- Y. Hu, D. Nualart, Renormalized self-intersection local time for fractional Brownian motion, *Ann. Probab.* 2005(33) 948-983.
- D. Nualart, G. Peccati, Central limit theorems for sequences of multiple stochastic integrals, *Ann. Probab.* 2005(33) 177-193.
- D. Nualart, S. Ortiz-Latorre, Central limit theorems for multiple stochastic integrals and Malliavin calculus, *Stoch. Proc. Appl.* 2008(118) 614-628.

G. Peccati, C.A. Tudor, Gaussian limits for vector-valued multiple stochastic integrals, in: M. Emery, M. Ledoux and M. Yor (eds), *Seminaire de Probabilites XXXVIII*, Lecture Notes in Math 1857, pp. 247-262. Springer-Verlag, Berlin 2005.