New central limit theorems for functionals of Gaussian processes and their applications

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Abstract

As a consequence of the seminal work of Nualart and Peccati in 2005 we have new central limit theorems for functional of Gaussian processes that have allowed us to elucidate the asymptotic behavior of the multipower variation of certain "ambit processes". This survey intends to explain the role of the Malliavin calculus to reach these results. It was presented in the workshop "Ambit processes, non-semimartingales and applications", held in Sandbjerg (Denmark) January 26th, 2010.

1 Introduction

Consider a sequence of Gaussian random variables $\{X_n, n \geq 1\}$ with $E(X_j) = 0$, $E(X_j^2) = 1$, $E(X_j X_k) = \rho(j,k)$, $j,k \geq 1$. Let $H(x)$ be a real value function such that, $E(H(X_1)) = 0$ and $E(H(X_1))^2 < \infty$. Then, a natural problem, is to find suitable conditions ensuring that

$$F_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} H(X_i) \underset{n\to\infty}{\to} N(0,1).$$

So far, the main way of solving this problem was to check if the moments of F_n converged to the moments of the standard normal distribution, that is, to see if

$$\lim_{n\to\infty} E(F_n^p) \underset{n\to\infty}{\to} \begin{cases} (p-1)!!, \text{ if } p \text{ is even} \\ 0 \text{ if } p \text{ is odd.} \end{cases}$$

To prove this one expanded $H(x)$ in the form

$$H(x) = \sum_{j=1}^{\infty} c_j H_j(x),$$

where H_j is the jth Hermite polynomial

$$H_j(x) = (-1)^n e^{x^2/2} \frac{d^j}{dx^j}(e^{-x^2/2}), \quad j \geq 1,$$

and by using the **diagram formula** to calculate the asymptotic moments of F_n.

Consider a set of vertices $\{(i,j), 1 \le i \le p, 1 \le j \le l_i\}$ and a diagram G with the following properties

1. Edges may pass only if the first coordinates of the vertices are different.

2. Each vertex has one edge.

Let $\Gamma = \Gamma(l_1, ..., l_n)$ denote the set of all these diagrams and for $G \in \Gamma$ by $A(G)$ the set of edges G. Now for a $w \in G$, $w = ((i_1, j_1), (i_2, j_2))$, where $i_1 < i_2$, define the functions $d_1(w) = i_1, d_2(w) = i_2$. Then

$$E(\Pi_{i=1}^p H_{l_i}(X_i)) = \sum_{G \in \Gamma} \Pi_{w \in A(G)} \rho(d_1(w), d_2(w)).$$

As a particular case we have

$$E(H_n(X_1)H_m(X_2)) = \delta_{nm} m! \rho^m(1,2),$$

where δ_{nm} is the Kronecker symbol.

Since the work of Nualart and Peccatti in (2005) we know that it is sufficient to check the behaviour of the second and the fourth order moments of (F_n), even we can get equivalent conditions for the convergence of the fourth order moments that are easier to check. Moreover the random variables F_n can be measurable with respect to a Gaussian process, not necessarily discrete.

The theoretical framework to obtain these results is the so called The Malliavin Calculus.

2 Basic Malliavin Calculus

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian subspace $\mathcal{H}_1$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero-mean Gaussian random variables. Let $\mathfrak{H}$ be a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and norm $|| \cdot ||_{\mathfrak{H}}$. We will assume that there is an isometry

$$\begin{aligned} W : \mathfrak{H} &\to \mathcal{H}_1 \\ h &\mapsto W(h) \end{aligned}$$

in the sense that

$$E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathfrak{H}}.$$

It is easy to see that this map has to be linear. W is called an isonormal Gaussian process.

Example 1. Let $\{e_i, i \ge 1\}$ be the canonical basis of $\mathbb{R}^{\mathbb{N}}$ with a scalar product $\langle e_i, e_j \rangle = \rho(i,j)$ consider $\mathfrak{H} = span\, \{e_i, i \ge 1\}$. Then $\{W(e_i), i \ge 1\}$ will be a sequence of centered Gaussian random variables with covariance function $\rho(\cdot, \cdot)$.

Example 2. Take $\langle \mathbf{1}_{[0,t]}(\cdot), \mathbf{1}_{[0,s]}(\cdot)\rangle = \rho(s,t)$, and $\mathfrak{H} = span\{\mathbf{1}_{[0,t]}(\cdot), 0 \le t \le T\}$ then $\left(W_t := W(\mathbf{1}_{[0,t]})\right)$is a centered Gaussian process with covariance function $\rho(\cdot,\cdot)$.

Example 3. If, in the previous example, we take $\rho(s,t) = s \wedge t$ then $\mathfrak{H} = L^2([0,T], dx)$ and $\left(W_t := W(\mathbf{1}_{[0,t]})\right)$ is a Brownian motion, moreover $W(h) = \int_0^T h_s dW_s$, the Wiener integral of the function h with respect to the Brownian motion (W_t) .

Example 4. $\mathfrak{H} = L^2(\mathbb{A}, \mathcal{A}, \mu)$ where $(\mathbb{A}, \mathcal{A}\ )$ is a measurable space and μ is a σ-finite measure without atoms (i.e. for any $A \in \mathcal{A}$ such that $\mu(A) > 0$ there is $B \in \mathcal{A}$ such that $0 < \mu(B) < \mu(A)$).

The process $\{W(A) := W(\mathbf{1}_A), A \in \mathcal{A}, \mu(A) < \infty\}$ is called a Gaussian white noise with intensity μ on the space $(\mathbb{A}, \mathcal{A}\ )$.

We can define a Wiener integral of a function $h \in \mathfrak{H}$ with respect to the process $(W(A))$ and we have that $W(h) = \int_{\mathbb{A}} h_s dW_s$. We can also construct, in a standard way, the multiple Wiener integral for functions in $L^2(\mathbb{A}^n, \mathcal{A}^n, \mu^n)$ and it can be seen that, if $h \in \mathfrak{H}$, $||h||_{\mathfrak{H}} = 1$, $H_n(W(h)) = \int_{\mathbb{A}^n} h(t_1)...h(t_n) dW_{t_1}...dW_{t_n}$.

For any $m \ge 2$, we denote by $\mathcal{H}_m$ the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $H_m(W(h))$, where $h \in \mathfrak{H}$, $||h||_{\mathfrak{H}} = 1$. It is called the m-th Wiener chaos. Then,

Theorem 5. *Every random variable* $Y \in L^2(\Omega, \mathcal{G}, P)$*, where* $\mathcal{G}$ *is the* σ*-field generated by* W*, can be uniquely expanded as*

$$Y = E(Y) + \sum_{n=1}^{\infty} Y_n,$$

where $Y_n \in \mathcal{H}_n$.

Proof. If $Y \in L^2(\Omega, \mathcal{G}, P)$ is orthogonal to every $H_n(W(h))$, $h \in \mathfrak{H}$, $||h||_{\mathfrak{H}} = 1$ then Y is orthogonal to $e^{\sum_{i=1}^n \lambda_i W(e_i)}$, $\lambda_i \in \mathbb{R}, i \ge 1$ and $(e_i)_{i\ge 1}$ an orthonormal basis of $\mathfrak{H}$. From here $E(Y|W(e_1), ...W(e_n)) = 0, a.s$ and since $E(Y|W(e_1), ...W(e_n))$ converges a.s. to Y, then $Y = 0, a.s.$. ■

Suppose that $\mathfrak{H}$ is infinite-dimensional and let $\{e_i, i \ge 1\}$ be an orthonormal basis of $\mathfrak{H}$. Denote by Λ the set of all sequences $a = (a_1, a_2, ...),\ a_i \in \mathbb{N}$, such that all the terms, except a finite number of them, vanish. For $a \in \Lambda$ we set $a! = \Pi_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multiindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

The family of random variables $\{\Phi_a, a \in \Lambda\}$ is an orthonormal system. In fact

$$E\left[\Pi_{i=1}^{\infty} H_{a_i}(W(e_i)) \Pi_{i=1}^{\infty} H_{b_i}(W(e_i))\right] = \delta_{ab} a!\ ,$$

Moreover, $\{\Phi_a |\ a \in \Lambda,\ |a| = m\}$ is a complete orthonormal system in $\mathcal{H}_m$.

Let $a \in \Lambda$ with $|a| = m$ and denote $\otimes_{i=1}^{\infty} e_i^{\otimes a_i} = e^{\otimes a}$. Where $\otimes$ is the tensor product. The mapping

$$\begin{array}{rcl} I_m : \mathfrak{H}^{\odot m} & \to & \mathcal{H}_m \\ \widetilde{e^{\otimes a}} & \mapsto & \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)), \end{array}$$

between the symmetric tensor product $\mathfrak{H}^{\odot m}$, equipped with the norm $\sqrt{m!}\,\|\cdot\|_{\mathfrak{H}^{\otimes m}}$, and the m-th chaos $\mathcal{H}_m$ is a linear isometry. Here $\widetilde{\otimes}$ denotes the symmetrization of the tensor product $\otimes$ and I_0 is the identity in $\mathbb{R}$.

For any $h = h_1 \otimes \cdots \otimes h_m$ and $g = g_1 \otimes \cdots \otimes g_m \in \mathfrak{H}^{\otimes m}$, we define the p-th contraction of h and g, denoted by $h \otimes_p g$, as the element of $\mathfrak{H}^{\otimes 2(m-p)}$ given by

$$h \otimes_p g = \langle h_1, g_1\rangle_{\mathfrak{H}} \cdots \langle h_p, g_p\rangle_{\mathfrak{H}} h_{p+1} \otimes \cdots \otimes h_m \otimes g_{p+1} \otimes \cdots \otimes g_m.$$

This definition can be extended by linearity to any element of $\mathfrak{H}^{\otimes m}$. $h \otimes_p g$ does not necessarily belong to $\mathfrak{H}^{\odot(2m-p)}$, even if h and g belong to $\mathfrak{H}^{\odot m}$. We denote by $h\widetilde{\otimes}_p g$ the symmetrization of $h \otimes_p g$.

Proposition 6. *For any $h \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, we have*

$$I_p(h)I_q(g) = \sum_{r=0}^{p\wedge q} r! \begin{pmatrix} p \\ r \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} I_{p+q-2r}(h\widetilde{\otimes}_r g).$$

Proof. First, note that

$$I_1(e_i) = W(e_i).$$

Let $a \in \Lambda$ with $|a| = p$ and $q = 1$. Due to linearity of I_p it suffices to consider the case $h = \widetilde{e^{\otimes a}}$, $g = e_j$. It holds that

$$I_p(\widetilde{e^{\otimes a}})I_1(e_j) = \Pi_{i=1}^{\infty} H_{a_i}(W(e_i))W(e_j).$$

Assume that j is an index such that $a_j = 0$. Then

$$\widetilde{e^{\otimes a}}\widetilde{\otimes}_1 e_j = 0$$

and

$$\Pi_{i=1}^{\infty} H_{a_i}(W(e_i))W(e_j) = I_{p+1}(\widetilde{e^{\otimes a}}\widetilde{\otimes} e_j),$$

so we have that

$$I_p(\widetilde{e^{\otimes a}})I_1(e_j) = I_{p+1}(\widetilde{e^{\otimes a}}\widetilde{\otimes} e_j) + pI_{p-1}(\widetilde{e^{\otimes a}}\widetilde{\otimes}_1 e_j).$$

Assume now that $a_j \neq 0$. Then we obtain the identity

$$\widetilde{e^{\otimes a}}\widetilde{\otimes}_1 e_j = \frac{a_j}{p}\widetilde{e^{\otimes a'(j)}}$$

with $a'_i(j) = a_i$ if $i \neq j$ and $a'_j(j) = a_j - 1$. Furthermore,since the Hermite polynomials verify

$$xH_n(x) = H_{n+1}(x) + nH_{n-1}(x).$$

we have that

$$\begin{aligned} & \Pi_{i=1}^{\infty} H_{a_i}(W(e_i))W(e_j) \\ = & \Pi_{i=1, i\neq j}^{\infty} H_{a_i}(W(e_i)(H_{a_j+1}(W(e_j)) + a_j H_{a_j-1}(W(e_j))) \\ = & I_{p+1}(\widetilde{e^{\otimes a}}\widetilde{\otimes} e_j) + pI_{p-1}(\widetilde{e^{\otimes a}}\widetilde{\otimes}_1 e_j), \end{aligned}$$

Hence, the multiplication formula is true for $q = 1$. The general formula follows by induction. ■

Theorem 7. *Let $h \in \mathfrak{H}$ with $||h||_{\mathfrak{H}} = 1$. Then for every $m \geq 1$ we have*

$$I_m(h^{\otimes m}) = H_m(W(h))$$

Proof. For $m = 1$ it is clear. then

$$\begin{aligned} I_{m+1}(h^{\otimes (m+1)}) &= I_m(h^{\otimes m})I_1(h) - mI_{m-1}(h^{\otimes (m-1)}) \\ &= H_m(W(h)W(h) - mH_{m-1}(W(h)) \\ &= H_{m+1} \end{aligned}$$

■

Theorem 8. *Every random variable $Y \in L^2(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is the σ-field generated by W, can be uniquely expanded as*

$$Y = \sum_{n=0}^{\infty} I_n(h_n),$$

where $h_n \in \mathfrak{H}^{\odot n}$.

Proof. It is immediate from the previous chaos decomposition and the definition of I_m. ■

Let $\mathcal{S}$ be the class of smooth random variables $F = f(W(h_1), ..., W(h_n))$, $f \in C_p^{\infty}(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), we can define its differential as

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), W(h_2), ..., W(h_n))h_i.$$

DF is as a random variable with values in $\mathfrak{H}$. Then we can built a closed map

$$\begin{aligned} D : \mathbb{D}^{1,2} \subseteq L^2(\Omega, \mathbb{R}) \quad &\longrightarrow \quad L^2(\Omega, \mathfrak{H}) \\ F \quad &\mapsto \quad DF. \end{aligned}$$

where $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables with respect to the norm

$$||F||_{1,2} = \left(E(|F|^2) + E(||DF||^2_{\mathfrak{H}})\right)^{1/2}.$$

For instance $D(H_n(W(h))) = nH_{n-1}(W(h))h, n \geq 1$, ($H_0 := 1$).

Proposition 9. *Set $h \in \mathfrak{H}^{\odot n}$, then*

$$E(\|DI_n(h)\|^2_{\mathfrak{H}}) = nn! \, \|h\|^2_{\mathfrak{H}^{\otimes n}}$$

Proof. It is sufficient to consider $h = h_1^{\otimes n}$, $h_1 \in \mathfrak{H}$. Then, $I_n(h_1^{\otimes n}) = ||h_1||^n_{\mathfrak{H}} H_n(W(h_1/||h_1||_{\mathfrak{H}}))$,

$$DI_n(h) = n||h_1||^{n-1}_{\mathfrak{H}} H_{n-1}(W(h_1/||h_1||_{\mathfrak{H}}))h_1,$$

and

$$\|DI_n(h)\|^2_{\mathfrak{H}} = n^2||h_1||^{2n}_{\mathfrak{H}} H_{n-1}(W(h_1/||h_1||_{\mathfrak{H}}))^2.$$

Therefore

$$E(\|DI_n(h)\|_{\mathfrak{H}}^2) = n^2||h_1||_{\mathfrak{H}}^{2n}(n-1)! = nn!||h||_{\mathfrak{H}^{\otimes n}}^2$$

■

Let u be an element of $L^2(\Omega, \mathfrak{H})$ and assume there is an element $\delta(u) \in L^2(\Omega)$ such that

$$E(\langle DF, u\rangle_{\mathfrak{H}}) = E(F\delta(u))$$

for any $F \in \mathbb{D}^{1,2}$, then we say that u is in the domain of δ and that δ is the adjoint operator of D. For instance

Proposition 10. *Let h be an element of $\mathfrak{H}$,*

$$\delta(h) = W(h)$$

Proof. Without loss of generality we can assume that $||h||_{\mathfrak{H}} = 1$ and that $F = f(W(h), W(h_2), ..., W(h_n))$ with h_i orthogonal to h. Then

$$\begin{aligned} & E(\langle DF, h\rangle_{\mathfrak{H}}) \\ = \; & E(\partial_1 f) = E(\int_{\mathbb{R}} \partial_1 f(x_1, W(h_2), ..., W(h_n)) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} dx_1) \\ = \; & E(\int_{\mathbb{R}} x_1 f(x_1, W(h_2), ..., W(h_n)) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} dx_1 \\ = \; & E(FW(h)). \end{aligned}$$

■

Proposition 11. *If*

$$u = \sum_{i=1}^{n} F_j h_j$$

where F_j are smooth random variables and h_j are elements of $\mathfrak{H}$ then

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j\rangle_{\mathfrak{H}}$$

Proof. Let $T \in \mathcal{S}$,

$$\begin{aligned} E(T\delta(u)) &= \sum_{j=1}^{n} E(TF_j W(h_j)) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}}) \\ &= \sum_{j=1}^{n} E(\langle D(TF_j), h_j\rangle_{\mathfrak{H}}) - \sum_{j=1}^{n} E(T\langle DF_j, h_j\rangle_{\mathfrak{H}}) \\ &= \sum_{j=1}^{n} E(\langle D(TF_j) - TDF_j, h_j\rangle_{\mathfrak{H}}) \\ &= \sum_{j=1}^{n} E(F_j\langle DT, h_j\rangle_{\mathfrak{H}}) = E(\langle DT, \sum_{j=1}^{n} F_j h_j\rangle_{\mathfrak{H}}) \\ &= E(\langle DT, u\rangle_{\mathfrak{H}}) \end{aligned}$$

■

Proposition 12. *If*

$$u = H_{n-1}(W(h))h$$

where $h \in \mathfrak{H}$, $\|h\|_{\mathfrak{H}} = 1$ *then*

$$\delta(u) = H_n(W(h)).$$

Proof.

$$\begin{aligned}
\delta(u) &= H_{n-1}(W(h))W(h) - \langle DH_{n-1}(W(h)), h\rangle_{\mathfrak{H}} \\
&= H_{n-1}(W(h))W(h) - (n-1)H_{n-2}(W(h))\langle h, h\rangle_{\mathfrak{H}} \\
&= H_{n-1}(W(h))W(h) - (n-1)H_{n-2}(W(h)) = H_n(W(h)).
\end{aligned}$$

■

Corollary 13. *Let* $F \in \mathcal{H}_n$ *then*

$$\delta DF = nF.$$

Proof. It is sufficient to consider F of the form $F = H_n(W(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1$. Then

$$\delta DF = \delta(nH_{n-1}(W(h))h) = nH_n(W(h)) = nF.$$

■

Lemma 14. *Consider two random variables* $F = I_n(f), G = I_m(g)$, *where* $n, m \geq 1$. *Then*

$$E(\langle DF, DG\rangle_{\mathfrak{H}}^2) = \sum_{r=1}^{n\wedge m} \frac{(n!m!)^2}{((n-r)!(m-r)!(r-1)!)^2} \left\| f\widetilde{\otimes}_r g\right\|^2_{\mathfrak{H}^{\odot(n+m-2r)}}.$$

Proof. It is sufficient to consider $f = \widetilde{e^{\otimes a}}$ and $g = \widetilde{e^{\otimes b}}$, then

$$\begin{aligned}
DI_n(f) &= \sum_{j=1}^{\infty} a_j I_{n-1}(\widetilde{e^{\otimes a'(j)}})e_j, \\
DI_m(g) &= \sum_{k=1}^{\infty} b_k I_{m-1}(\widetilde{e^{\otimes b'(k)}})e_k,
\end{aligned}$$

and

$$\begin{aligned}
&\langle DI_n(f), DI_m(g)\rangle_{\mathfrak{H}} \\
&= \sum_{k=1}^{\infty} a_k b_k I_{n-1}(\widetilde{e^{\otimes a'(j)}}) I_{m-1}(\widetilde{e^{\otimes b'(k)}}) \\
&= \sum_{k=1}^{\infty} a_k b_k \sum_{r=0}^{m\wedge n-1} r! \begin{pmatrix} n-1 \\ r \end{pmatrix} \begin{pmatrix} m-1 \\ r \end{pmatrix} I_{m+n-2-2r}(\widetilde{e^{\otimes a'(k)}} \widetilde{\otimes}_r \widetilde{e^{\otimes b'(k)}}).
\end{aligned}$$

Hence,

$$\begin{aligned} & \langle DI_n(f), DI_m(g)\rangle_{\mathfrak{H}} \\ = & \sum_{r=0}^{m\wedge n-1} r! \begin{pmatrix} n-1 \\ r \end{pmatrix} \begin{pmatrix} m-1 \\ r \end{pmatrix} I_{m+n-2-2r}\Big(\sum_{k=1}^{\infty} a_k b_k \widetilde{e^{\otimes a'(k)}} \widetilde{\otimes}_r \widetilde{e^{\otimes b'(k)}}\Big) \\ = & \; nm \sum_{r=0}^{m\wedge n-1} r! \begin{pmatrix} n-1 \\ r \end{pmatrix} \begin{pmatrix} m-1 \\ r \end{pmatrix} I_{m+n-2-2r}(\widetilde{e^{\otimes a}} \widetilde{\otimes}_{r+1} \widetilde{e^{\otimes b}}). \end{aligned}$$

Finally

$$\begin{aligned} & E\left(\langle DI_n(f), DI_m(g)\rangle_{\mathfrak{H}}\right)^2 \\ = & \; (nm)^2 \sum_{r=0}^{m\wedge n-1} (r!)^2 \begin{pmatrix} n-1 \\ r \end{pmatrix}^2 \begin{pmatrix} m-1 \\ r \end{pmatrix}^2 \left\| \widetilde{e^{\otimes a}} \widetilde{\otimes}_{r+1} \widetilde{e^{\otimes b}} \right\|^2_{\mathfrak{H}^{\odot(n+m-2-2r)}} \end{aligned}$$

■

3 Central limit theorems of random variables

Theorem 15. *Fix $n \geq 2$. Consider a sequence $\{F_k = I_n(f_k), k \geq 1\}$ such that*

$$E(F_k^2) \underset{k\to\infty}{\to} \sigma^2 \tag{1}$$

The following statements are equivalent:

(i) $F_k \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} N(0,\sigma^2)$.

(ii) $E(F_k^4) \underset{k\to\infty}{\to} 3\sigma^4$.

(iii) $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(n-r)}} \underset{k\to\infty}{\to} 0$, *for all* $1 \leq r \leq n-1$.

(iv) $\|DF_k\|^2_{\mathfrak{H}} \underset{k\to\infty}{\overset{L^2(\Omega)}{\to}} n\sigma^2$.

Proof. To simplify we take $\sigma^2 = 1$. We shall prove the following implications

$$(iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

$(iv) \Rightarrow (i)$. First the sequence (F_k) is tight since it is bounded in $L^2(\Omega)$ by condition (1).Second, assume that F_k converges to G. Again by (1), $G \in L^2(\Omega)$. Then its characteristic function $\varphi(t) = E(e^{itG})$ is differentiable and $\varphi'(t) = iE(Ge^{itG})$. For every $k \geq 1$, define $\varphi_k(t) = E(e^{itF_k})$, then $\varphi_k'(t) = iE(F_k e^{itF_k})$. Clearly, $F_k e^{itF_k}$ converges in law to Ge^{itG} and the boundedness in $L^2(\Omega)$, implies convergence of the first order moments.

$(iv) \Rightarrow (i)(cont.)$. Then, $\varphi_k'(t) \to \varphi'(t)$. Moreover,

$$\begin{aligned}\varphi_k'(t) &= iE(F_k e^{itF_k}) = \frac{i}{n}E(\delta D(F_k)e^{itF_k}) \\ &= -\frac{t}{n}E(e^{itF_k}\langle DF_k, DF_k\rangle_{\mathfrak{H}}) \\ &= -\frac{t}{n}E(e^{itF_k}\|DF_k\|_{\mathfrak{H}}^2) \to -t\varphi(t),\end{aligned}$$

in fact, by (iv)

$$\left|E(e^{itF_k}\|DF_k\|_{\mathfrak{H}}^2) - n\varphi(t)\right| \le E(\left|\|DF_k\|_{\mathfrak{H}}^2 - n\right|) + n\left|E(e^{itF_k}) - \varphi(t)\right|.$$

This implies that $\varphi(t)$ satisfies the differential equation

$$\varphi'(t) = -t\varphi(t), \varphi(0) = 1.$$

$(i) \Rightarrow (ii)$. It is well known that, for any $1 < p < q < \infty$ the norms $\|\cdot\|_p, \|\cdot\|_q$ are equivalent in any Wiener chaos $\mathcal{H}_n$, then convergence in law and convergence of the second order moments implies convergence of the moments of any order.

$(ii) \Rightarrow (iii)$.By using the product formula

$$\begin{aligned}I_n(f_k)^2 &= \sum_{r=0}^{n} r!\begin{pmatrix} n \\ r\end{pmatrix}^2 I_{2(n-r)}(f_k\widetilde{\otimes}_r f_k) \\ &= n!||f_k||^2 + I_{2n}\left(f_k\widetilde{\otimes}f_k\right) + \sum_{r=1}^{n-1} r!\begin{pmatrix} n \\ r\end{pmatrix}^2 I_{2(n-r)}(f_k\widetilde{\otimes}_r f_k).\end{aligned}$$

Then

$$\begin{aligned}&E(I_n(f_k)^4) = (n!)^2\,||f_k||^4 \\ &+(2n)!\left\|f_k\widetilde{\otimes}f_k\right\|_{\mathfrak{H}^{\otimes 2n}}^2 + \sum_{r=1}^{n-1}(r!)^2\begin{pmatrix} n \\ r\end{pmatrix}^4 (2(n-r))!\left\|f_k\widetilde{\otimes}_r f_k\right\|_{\mathfrak{H}^{\otimes 2(n-r)}}^2,\end{aligned}$$

also we have that

$$||f_k\widetilde{\otimes}f_k||_{\mathfrak{H}^{\otimes 2n}}^2 = f_k \otimes f_k \otimes_{2n} f_k\widetilde{\otimes}f_k.$$

$(ii) \Rightarrow (iii)$. Therefore, $\left\|f_k\widetilde{\otimes}f_k\right\|_{\mathfrak{H}^{\otimes 2n}}^2$ is the sum of $\frac{(2n)!}{(n!)^2}$ terms of the form $\frac{(n!)^2}{(2n)!}||f_k \otimes_a f_k||^2$ with $a = 0, 1, .., n$. And for $a = 0, n$

$$||f_k \otimes_a f_k||_{\mathfrak{H}^{\otimes 2(n-a)}}^2 = ||f_k||^4.$$

Consequently

$$E(I_n(f_k)^4) = 3\,(n!)^2\,||f_k||^4 + R_k,$$

and, by the hypothesis (ii) $R_k \to 0$, equivalently $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(n-r)}}^2 \to 0, 1 \le r \le n-1$.

$(iii) \Rightarrow (iv)$. We know that $E(\|DI_n(h)\|^2_{\mathfrak{H}}) = nn!\,\|h\|^2_{\mathfrak{H}}$, then

$$E((\|DF_k\|^2_{\mathfrak{H}} - n)^2) = E(\|DF_k\|^4_{\mathfrak{H}}) - 2n^2 n!\,\|f_k\|^2_{\mathfrak{H}^{\otimes n}} + n^2,$$

therefore it suffices to prove that (iii) implies that $E(\|DF_k\|^4_{\mathfrak{H}}) \to n^2$. But, by the previous Lemma we have that

$$E(\|DF_k\|^4_{\mathfrak{H}}) = \sum_{r=1}^{n-1} \frac{(n!)^4}{\left(((n-r)!)^2\,(r-1)!\right)^2} \left\|f_k \widetilde{\otimes}_r f_k\right\|_{\mathfrak{H}^{\odot 2(n-r)}} + n^2 (n!)^2\,\|f_k\|^4_{\mathfrak{H}^{\otimes n}},$$

so, by (iii), $E(\|DF_k\|^4_{\mathfrak{H}}) \to n^2$. ∎

Example

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\mathfrak{H} = L^2([0,1], dx)$. Then

$$F_k := \sqrt{k}\left(\frac{1}{k}\int_0^1 B_t^2 t^{1/k-2} dt - 1\right) \underset{k\to\infty}{\to} N(0,2).$$

It is easy to see that

$$\begin{aligned} F_k &= \frac{\sqrt{k}}{k-1}\int_0^1\int_0^1 \left((s\vee t)^{1/k-1} - 1\right) dB_s dB_t \\ &= I_2(f_k), \end{aligned}$$

where

$$f_k = \frac{\sqrt{k}}{k-1}\left((s\vee t)^{1/k-1} - 1\right) \in \mathfrak{H}^{\odot 2}$$

We have that

$$\begin{aligned} E(F_k^2) &= 2||f_k||^2_{\mathfrak{H}^{\otimes 2}} = \frac{2k}{(k-1)^2}\int_0^1\int_0^1 \left((s\vee t)^{1/k-1} - 1\right) ds dt \\ &= \frac{4k}{(k-1)^2}\left(\frac{k}{2} - \frac{2k}{k+1} + \frac{1}{2}\right) \underset{k\to\infty}{\to} 2. \end{aligned}$$

Then it is enough to see that

$$||f_k \otimes_1 f_k||^2_{\mathfrak{H}^{\otimes 2}} \underset{k\to\infty}{\to} 0.$$

But

$$\begin{aligned} & ||f_k \otimes_1 f_k||^2_{\mathfrak{H}^{\otimes 2}} \\ = \; & \frac{k^2}{(k-1)^4}\int_0^1\int_0^1 \left(\int_0^1 \left((s\vee t)^{1/k-1} - 1\right)\left((s\vee u)^{1/k-1} - 1\right) ds\right)^2 dt du, \end{aligned}$$

and $\int_0^1\int_0^1 \left(\int_0^1 \left((s\vee t)^{1/k-1} - 1\right)\left((s\vee u)^{1/k-1} - 1\right) ds\right)^2 dt du = O(k)$.

Example

Consider a sequence of stationary, normalized, centered Gaussian random variables $(X_i)_{i\geq 1}$. We want to study the asymptotic behavior of the sequence

$$F_k := \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) \; ,$$

$m \geq 2$. We can take $\mathcal{H}_1 = span\{X_i, i \geq 1\}$, and $\mathfrak{H} \equiv \mathcal{H}_1$. The inner product on $\mathfrak{H}$ is then induced by the covariance function $\rho(k) = \mathrm{cov}(X_1, X_{1+k})$ of the sequence $(X_i)_{i\geq 1}$ (note that $\rho(0) = 1$).We obtain the following representation

$$F_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} H_m(X_i) = I_m\left(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i^{\otimes m}\right).$$

Set

$$h_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i^{\otimes m}.$$

Assume that

$$\sum_{j=1}^{\infty} |\rho(j)|^m < \infty. \tag{2}$$

It holds that

$$\begin{aligned} m!||h_k||^2_{\mathfrak{H}^{\otimes m}} &= \frac{m!}{k} \sum_{i,j=1}^{k} (E(X_i X_j))^m = \frac{m!}{k} \sum_{i,j=1}^{k} \rho^m(i-j) \\ &= m!\left(1 + 2\sum_{j=1}^{k-1} \rho^m(j)\left(1 - \frac{j}{k}\right)\right) \to m!\left(1 + 2\sum_{j=1}^{\infty} \rho^m(j)\right) =: \sigma^2. \end{aligned}$$

Note the identity

$$h_k \otimes_r h_k = \frac{1}{k} \sum_{i,j=1}^{k} \rho^r(i-j) X_i^{\otimes(m-r)} \otimes X_j^{\otimes(m-r)},$$

This implies

$$
\begin{aligned}
& \|h_k \otimes_r h_k\|^2_{\mathfrak{H}^{\otimes 2(m-r)}} \\
= \;& \frac{1}{k^2} \sum_{i,j,i',j'=1}^{k} \rho^r(i-j)\rho^r(i'-j')\rho^{m-r}(i-i')\rho^{m-r}(j-j') \\
= \;& \frac{1}{k} \sum_{i,j,i'=0}^{k-1} \rho^r(i)\rho^r(j-i')\rho^{m-r}(j)\rho^{m-r}(i-i')(1-\frac{i\vee j\vee i'}{k}) \\
\leq \;& \frac{1}{k} \sum_{i,j,i'=0}^{k-1} \rho(i)\rho(j-i')\rho(j)\rho(i-i') \\
= \;& \frac{1}{k} \sum_{i=0}^{k-1} \left(\sum_{j=0}^{k-1} \rho(j)\rho(i-j) \right)^2 \leq 2\varepsilon \left(\sum_{j=0}^{\infty} \rho(j)^2 \right)^2,
\end{aligned}
$$

for any $\varepsilon > 0$. So the last term converges to 0 under assumption (2) for $1 \leq r \leq m-1$, and we deduce that $F_k \xrightarrow[k\to\infty]{\mathcal{L}} N(0,\sigma^2)$.

For $d \geq 2$, fix d natural numbers, $1 \leq n_1 \leq ... \leq n_d$. Consider a sequence of random vectors

$$F_k = (F_k^1, .., F_k^d) = (I_{n_1}(f_k^1), ..., I_{n_d}(f_k^d)), \tag{3}$$

where $f_k^i \in \mathfrak{H}^{\odot n_i}$. We have a multidimensional version of the previous theorem,

Theorem 16. *Let $(F_k)_{k\geq 1}$ be a sequence of random vectors of the form (3) such that, for every $1 \leq i, j \leq d$*

$$\lim E(F_k^i F_k^j) = \delta_{ij}, \tag{4}$$

then the following statements are equivalent

(i) For every $i = 1, ..., d$, $F_k^i \xrightarrow[k\to\infty]{\mathcal{L}} N(0,1)$.

(ii) For every $i = 1, ..., d$, $E(\left(F_k^i\right)^4) \xrightarrow[k\to\infty]{} 3$.

(iii) $\left\|f_k^i \otimes_r f_k^i\right\|_{\mathfrak{H}^{\otimes 2(n_i - r)}} \xrightarrow[k\to\infty]{} 0$, for all $1 \leq r \leq n_i - 1, 1 \leq i \leq d$.

(iv) For every $i = 1, ..., d$, $\left\|DF_k^i\right\|^2_{\mathfrak{H}} \xrightarrow[k\to\infty]{L^2(\Omega)} n_i$.

(v) $F_k \xrightarrow[k\to\infty]{\mathcal{L}} N_d(0, I_d)$

Proof.
$(iv) \Rightarrow (v)$. Assume then that F_k converges to G and that $\varphi(t) = E(e^{i\langle t,G\rangle})$. Set

$\varphi_k(t) = E(e^{i\langle t,F_k\rangle})$, then $\partial_j\varphi_k(t) = iE(F_k^j e^{i\langle t,F_k\rangle}) \to \partial_j\varphi(t)$. Moreover, by (iv),

$$\begin{aligned}\partial_j\varphi_k(t) &= iE(F_k^j e^{i\langle t,F_k\rangle}) = \frac{i}{n_j}E(\delta D(F_k^j)e^{i\langle t,F_k\rangle}) \\ &= -\frac{1}{n_j}\sum_{l=1}^{d} t_l E(e^{i\langle t,F_k\rangle}\langle DF_k^j, DF_k^l\rangle_{\mathfrak{H}}) \\ &= -\frac{t_j}{n_j}E(e^{i\langle t,F_k\rangle}\left\|DF_k^j\right\|_{\mathfrak{H}}^2) \to -t_j\varphi(t),\end{aligned}$$

in fact, for $j \neq l$, by using the Lemma and condition (4) it is easy to see that

$$E(\langle DF_k^j, DF_k^l\rangle_{\mathfrak{H}}^2) \to 0.$$

This implies that $\varphi(t)$ satisfies the partial differential equation

$$\begin{aligned}\partial_j\varphi(t) &= -t_j\varphi(t),\ j = 1, ..., d. \\ \varphi(0) &= 1.\end{aligned}$$

■

Finally, we can consider a d-dimensional random vector $F_k = (Y_k^1, \ldots, Y_k^d)^T$ which has a chaos representation

$$F_k^i = \sum_{m=1}^{\infty} I_m(f_{m,k}^i)\ , \qquad i = 1, \ldots, d\ ,$$

with $f_{m,k}^i \in \mathfrak{H}^{\odot m}$.

Theorem 17. *Suppose that the following conditions hold:*

(i) *For any* $i = 1, \ldots, d$ *we have* $\sum_{m=1}^{\infty} \sup_k m! ||f_{m,k}^i||_{\mathfrak{H}^{\otimes m}}^2 < \infty$.

(ii) *For any* $m \geq 1$, $i, j = 1, \ldots, d$ *we have constants* Σ_{ij}^m *such that*

$$\lim_{k\to\infty} E[I_m(f_{m,k}^i)I_m(f_{m,k}^j)] = \lim_{k\to\infty}\left\langle f_{m,k}^i, f_{m,k}^j\right\rangle_{\mathfrak{H}^{\odot m}} = \Sigma_{ij}^m\ ,$$

and the matrix $\Sigma^m = (\Sigma_{ij}^m)_{1\leq i,j\leq d}$ *is positive definite for all* m.

(iii) $\sum_{m=1}^{\infty}\Sigma^m = \Sigma \in \mathbb{R}^{d\times d}$.

(iv) *For any* $m \geq 1$, $i = 1, \ldots, d$ *and* $p = 1, \ldots, m-1$

$$\lim_{k\to\infty} ||f_{m,k}^i \otimes_p f_{m,k}^i||_{\mathfrak{H}^{\otimes 2(m-p)}}^2 = 0.$$

Then we have $F_k \xrightarrow[k\to\infty]{\mathcal{L}} N_d(0,\Sigma)$.

Proof. Fix $\upsilon \in \mathbb{R}^d$. By the theorem for the unidimensional case and condition (ii) and (iv) we have that $I_m(\upsilon^T f_{m,k})$ converges to a $N(0, \upsilon^T \Sigma^m \upsilon)$ as k goes to infinity. Then if use the theorem for the multidimensional case

$$\big(I_1(\upsilon^T f_{1,k}), ..., I_m(\upsilon^T f_{m,k})\big) \xrightarrow[k\to\infty]{\mathcal{L}} (\xi_1, ..., \xi_m), \tag{5}$$

where, for $i \geq 1$, ξ_i are independent $N(0, \upsilon^T \Sigma^i \upsilon)$. Define for every $N \geq 1$,

$$\begin{aligned} F_k^N &= \sum_{m=1}^{N} I_m(f_{m,k}) \,, \\ \xi^N &= \sum_{m=1}^{N} \xi_m \end{aligned}$$

Set also $\xi = \sum_{m=1}^{\infty} \xi_m$. Let $f \in C^1$ bounded and with bounded derivative, then

$$\begin{aligned} \big|E(f(\upsilon^T F_k) - f(\xi))\big| &\leq \big|E(f(\upsilon^T F_k) - f(\upsilon^T F_k^N))\big| \\ &\quad + \Big|E(f(\upsilon^T F_k^N) - f(\xi^N))\Big| + \Big|E(f(\xi) - f(\xi^N))\Big| \\ &\leq C|\upsilon| \left(\sum_{m=N+1}^{\infty} E(I_m(f_{m,k})^2) \right)^{1/2} \\ &\quad + \Big|E(f(\upsilon^T F_k^N) - f(\xi^N))\Big| + \Big|E(f(\xi) - f(\xi^N))\Big| . \end{aligned}$$

So, by conditions (i), (5) and (iii), if we take the supremum in k and then the limit in N we obtain the result. ■

4 Functional central limit theorems. The power and multipower variation

Let $(G_t)_{t\geq 0}$ be a Gaussian process which has centered and stationary increments. We want to study the asymptotic properties of the process

$$V(G,p)_t^n = \frac{1}{n\tau_n^p} \sum_{i=1}^{[nt]} |\Delta_i^n G|^p \,,$$

where $\Delta_i^n G = G_{\frac{i}{n}} - G_{\frac{i-1}{n}}$, $\tau_n^2 = E[|\Delta_i^n G|^2]$ and $p > 0$. Write

$$r_n(j) = \mathrm{Cov}\Big(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\Big) \,, \qquad j \geq 0.$$

and assume that

$$|r_n(j)|^2 \leq C j^{-1-\varepsilon}, \; j \geq 0, \text{for some } \varepsilon > 0 \tag{6}$$

and

$$\lim_{n\to\infty} r_n(j) = \rho(j),$$

Set $H(x) = |x|^p - \mu_p$, where $\mu_p = E(|N(0,1)|^p)$, then $H(x) = \sum_{j=2}^{\infty} a_j H_j(x)$, with $a_2 > 0$ and we have the following theorem:

Theorem 18.

$$\left(G_t, \sqrt{n}(V(G,p)_t^n - t\mu_p)\right) \xrightarrow[n\to\infty]{\mathcal{L}} \left(G_t, \sigma W_t\right),$$

where W is a Brownian motion that is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, independent of G and σ^2 is given by

$$\sigma^2 = \sum_{m=2}^{\infty} \sigma_m^2, \quad \sigma_m^2 = m! a_m^2 \lambda_m^2\,, \quad \lambda_m^2 = 1 + 2\sum_{i=1}^{\infty} \rho^m(i). \tag{7}$$

Proof. First we have to show the convergence of the f.d.d. Let $(a_k, b_k]$ pairwise disjoint intervals in $[0,T]$. Define

$$\begin{aligned} G_n^k &= \tau_n \sum_{i=[na_k]+1}^{[nb_k]} \frac{\Delta_i^n G}{\tau_n}, \\ Y_n^k &= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big)\,, \end{aligned}$$

it suffices to prove that

$$(G_n^k, Y_n^k)_{1\leq k\leq d} \xrightarrow{\mathcal{L}} \Big(G_{b_k} - G_{a_k}, \sigma(W_{b_k} - W_{a_k})\Big)_{1\leq k\leq d}\,,$$

where σ is given by (7) and W is independent of G.

Let $\mathcal{H}_1$ the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nT]}$. Notice that $\mathcal{H}_1$ is a separable Hilbert space with the scalar product induced by the covariance function of the triangular array $(\Delta_j^n G/\tau_n)_{n\geq 1, 1\leq j\leq [nT]}$. Then we can take $\mathfrak{H} = \mathcal{H}_1$ and try to apply the general CLT to this case.

We have

$$\begin{aligned} Y_n^k &= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big) \\ &= \frac{1}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \sum_{m=2}^{\infty} a_m H_m\left(\frac{\Delta_i^n G}{\tau_n}\right) \\ &= \sum_{m=2}^{\infty} I_m\left(\frac{a_m}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left(\frac{\Delta_i^n G}{\tau_n}\right)^{\otimes m}\right), \end{aligned}$$

and

$$G_n^k = \tau_n \sum_{i=[na_k]+1}^{[nb_k]} \frac{\Delta_i^n G}{\tau_n} = I_1\left(\tau_n \sum_{i=[na_k]+1}^{[nb_k]} \frac{\Delta_i^n G}{\tau_n}\right),$$

The components $\left(Y_n^k\right)$ and $\left(G_n^k\right)$ are orthogonals and it is clear that $\left(G_n^k\right) \underset{n\to\infty}{\overset{a.s.}{\to}} (G_{b_k} - G_{a_k})$.So we have just to prove that $\left(Y_n^k\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N_d(0,\sigma^2 I_d)$.

Then we can apply the previous theorem with

$$f_{m,n}^k = \frac{a_m}{\sqrt{n}} \sum_{i=[na_k]+1}^{[nb_k]} \left(\frac{\Delta_i^n G}{\tau_n}\right)^{\otimes m} \in \mathfrak{H}^{\odot m}.$$

Proof (i). Take, by simplicity, $k=1, a_1=0, b_1=1$

$$\begin{aligned} \|f_{m,n}^1\|_{\mathfrak{H}^{\otimes m}}^2 &= \frac{a_m^2 m!}{n} \sum_{i=1}^{n}\sum_{j=1}^{n} r_n^m(|i-j|) \\ &= a_m^2 m!(1+2\sum_{i=1}^{n-1}(1-\frac{i}{n})r_n^m(i)) \qquad (8) \\ &\underset{n\to\infty}{\to} a_m^2 m!(1+2\sum_{i=1}^{\infty}\rho^m(i)) = \sigma_m^2. \qquad (9) \end{aligned}$$

Proof (ii) and (iii). Take again by simplicity and w.l.o.g $k=1, a_1=0, b_1=a_1=1$ and $b_2=2$

$$\begin{aligned} |\left\langle f_{m,n}^1, f_{m,n}^2\right\rangle_{\mathfrak{H}^{\odot m}}| &= \left|\frac{a_m^2 m!}{n}\sum_{i=1}^{n}\sum_{j=n+1}^{2n} r_n^m(j-i)\right| \\ &= \left|\frac{a_m^2 m!}{n}\left(\sum_{j=1}^{n} j r_n^m(j) + \sum_{j=1}^{n-1} j r_n^m(2n-j)\right)\right| \\ &\leq \frac{a_m^2 m!}{n}\left(\sum_{j=1}^{n} j r^m(j) + \sum_{j=1}^{n-1} j r^m(2n-j)\right) \\ &\underset{n\to\infty}{\to} 0 \text{ (since } n r^m(n) \underset{n\to\infty}{\to} 0). \end{aligned}$$

Proof (iv). Fix $1 \leq p \leq m-1$.

$$f_{m,n}^1 \otimes_p f_{m,n}^1 = \frac{1}{n}\sum_{i,j=1}^{n} r_n^p(|i-j|)\left(\frac{\Delta_i^n G}{\tau_n}\right)^{\otimes(m-r)} \otimes_p \left(\frac{\Delta_j^n G}{\tau_n}\right)^{\otimes(m-r)},$$

and we have, like in the second example, that

$$\left\|f_{m,n}^1 \otimes_p f_{m,n}^1\right\|^2 \leq 2\varepsilon\left(\sum_{l=1}^{\infty}\rho(l)^2\right)^2.$$

Proof (tightness). The second step of the proof is to check the tightness condition of $(G_t, Z_t^{(n)})$ where

$$Z_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} H\Big(\frac{\Delta_i^n G}{\tau_n}\Big) .$$

Set

$$Z_t^{n,N} := \sum_{m=2}^{N} I_m\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m}\Big).$$

then we have, for $s < t$,

$$\begin{aligned} & E(|Z_t^{n,N} - Z_s^{n,N}|^2) \\ = \; & E(\sum_{m=2}^{N} I_m\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-[ns]} \Big(\frac{\Delta_i^n G}{\tau_n}\Big)^{\otimes m}\Big)^2) \\ = \; & \frac{[nt]-[ns]}{n} \sum_{m=2}^{N} \frac{1}{[nt]-[ns]} \sum_{i=1}^{[nt]-[ns]} \sum_{j=1}^{[nt]-[ns]} r_n^m(|i-j|) \\ \leq \; & C \frac{[nt]-[ns]}{n}. \end{aligned}$$

By the equivalence of the L^p norms for $1 < p < \infty$ on a fixed sum of Wiener chaos,

$$E(|Z_t^{n,N} - Z_s^{n,N}|^4)^{1/2} \leq C \frac{[nt]-[ns]}{n}$$

Then by the Cauchy-Schwarz inequality we obtain the approximation

$$\begin{aligned} & P\Big(|Z_t^{n,N} - Z_{t_1}^{n,N}| \geq \lambda, |Z_{t_2}^{n,N} - Z_t^{n,N}| \geq \lambda\Big) \\ \leq \; & C \frac{([nt]-[nt_1])([nt_2]-[nt])}{n^2 \lambda^4} \leq C \frac{(t_2-t_1)^2}{\lambda^4} \end{aligned}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$.

Moreover we have proved in the first step, by (9) and (6), that

$$\lim_{N\to\infty} \sup_n E[|Z_t^n - Z_t^{n,N}|^2] = 0.$$

Using this we conclude that

$$P\Big(|Z_t^n - Z_{t_1}^n| \geq \lambda, |Z_{t_2}^n - Z_t^n| \geq \lambda\Big) \leq C \frac{(t_2-t_1)^2}{\lambda^4}$$

for any $t_1 \leq t \leq t_2$ and $\lambda > 0$, from which we deduce the tightness of the sequence Z_t^n by Billingsley's criterium. ■

If we want to study the asymptotic behaviour of the *bipower variation processes*

$$V(G;p,q)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]} \left|\frac{\Delta_i^n G}{\tau_n}\right|^p \left|\frac{\Delta_{i+1}^n G}{\tau_n}\right|^q , \qquad p,q \geq 0 ,$$

we can consider

$$Z_t^n := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left(\left| \frac{\Delta_i^n G}{\tau_n} \right|^p \left| \frac{\Delta_{i+1}^n G}{\tau_n} \right|^q - \mu_{p,q}^{(n)} \right) , \qquad p, q \geq 0 ,$$

where $\mu_{p,q}^{(n)} := E \left(\left| \frac{\Delta_i^n G}{\tau_n} \right|^p \left| \frac{\Delta_{i+1}^n G}{\tau_n} \right|^q \right)$.

Then by using the product formula we have

$$Z_t^n = \sum_{m=2}^{\infty} I_m \Big(\frac{1}{n} \sum_{i=1}^{[nt]} f_{m,n}^i \Big),$$

where

$$f_{m,n}^i = \sum_{h=0}^{m} s_{h,m}^{(n)} \Big(\frac{\Delta_i^n G}{\tau_n} \Big)^{\otimes h} \widetilde{\otimes} \Big(\frac{\Delta_{i+1}^n G}{\tau_n} \Big)^{\otimes m-h}$$

and

$$s_{h,m}^{(n)} = \sum_{l=0}^{\infty} a_{p,l+h} a_{q,l+m-h} l! \left(\begin{array}{c} l+h \\ l \end{array} \right) \left(\begin{array}{c} l+m-h \\ l \end{array} \right) r_n^l(1).$$

Now we can introduce two independent variables $X_i^n(1), X_i^n(2) \sim N(0,1)$ that are given by

$$X_i^n(1) = \frac{\Delta_i^n G}{\tau_n} , \qquad X_i^n(2) = a_n \frac{\Delta_i^n G}{\tau_n} + b_n \frac{\Delta_{i+1}^n G}{\tau_n}$$

with $b_n = (1 - r_n^2(1))^{-1/2}$ and $a_n = -(1/r_n^2(1) - 1)^{-1/2}$.

It is clear that $f_{m,n}^i$ can be represented as

$$f_{m,n}^i = \sum_{k_l \in \{1,2\}} c_{k_1,\ldots,k_m}^n X_i^n(k_1) \otimes \cdots \otimes X_i^n(k_m) ,$$

for some constants $c_{k_1,\ldots,k_m}^n$. Note that all summands are orthogonal. We obtain

$$||f_{m,n}^i||^2_{\mathfrak{H}_1^{\otimes m}} = \sum_{k_l \in \{1,2\}} |c_{k_1,\ldots,k_m}^n|^2 =: c_m^n.$$

Also we have that

$$\begin{aligned} & |\langle f_{m,n}^1, f_{m,n}^{1+k} \rangle_{\mathfrak{H}^{\otimes m}}| \\ = & \sum_{h_l \in \{1,2\}, g_l \in \{1,2\}} c_{h_1,\ldots,h_m}^n c_{g_1,\ldots,g_m}^n \prod_{l=1}^{m} \langle X_i^n(h_l), X_{i+k}^n(g_l) \rangle_{\mathcal{H}_1} \\ \leq & \; c_m^n (C r(k-1))^m. \end{aligned}$$

And by using these results we can prove the central limit theorem for $V(G; p, q)_t^n$.

A similar extension works for the multipower variation

$$V(G,p_1,\ldots,p_k)^n_t = \frac{1}{n}\sum_{i=1}^{[nt]-k+1}\prod_{j=1}^{k}\left|\frac{\Delta^n_{i+j-1}G}{\tau_n}\right|^{p_j} \quad , \quad p_1,\ldots,p_k \geq 0\ ,$$

and for the **joint** multipower variation:

$$\big(V(G,p^1_1,\ldots,p^1_k)^n_t, ..., V(G,p^d_1,\ldots,p^d_k)^n_t\big)\ .$$

Define

$$\rho^{(n)}_{p_1,\ldots,p_k} = E\Big[\Big|\frac{\Delta^n_1 G}{\tau_n}\Big|^{p_1}\cdots\Big|\frac{\Delta^n_k G}{\tau_n}\Big|^{p_k}\Big].$$

We have

Theorem 19.

$$\bigg(G_t, \sqrt{n}\Big(V(p^j_1,\ldots,p^j_k)^n_t - \rho^{(n)}_{p^j_1,\ldots,p^j_k}t\Big)_{1\leq j\leq d}\bigg) \to (G_t, \beta^{1/2}W_t),$$

where W is a d-dimensional Brownian, defined in an extension of the original filtered space, independent of G, β is a $d\times d$-dimensional matrix given by

$$\beta_{ij} = \lim_{n\to\infty} n\, cov\Big(V_Q(p^i_1,\ldots,p^i_k)^n_1, V_Q(p^j_1,\ldots,p^j_k)^n_1\Big)\ , \quad 1\leq i,j\leq d,$$

and $(Q_i)_{i\geq 1}$ is stationary centered discrete time Gaussian process with correlation function $\rho(j)$.

References

[1] Barndorff-Nielsen, O.E., Corcuera, J.M. and Podolskij, M. (2009a): Power variation for Gaussian processes with stationary increments, *Stoch. Proc. Appl.* **119**, 1845-1865.

[2] Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. and Woerner, J.H.C. (2009b): Bipower variation for Gaussian processes with stationary increments, *J. Appl. Prob.* **46**, 132-150.

[3] Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. (2009c): Multipower variation for Brownian Semistationary processes. Preprint IMUB 412.

[4] Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. (2009d): Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. Preprint IMUB 413.

[5] Corcuera, J.M. Nualart, D. and Woerner, J.H.C. (2006): Power variation of some integral fractional processes. *Bernoulli* **12**, 713-735.

[6] Y. Hu, D. Nualart, Renormalized self-intersection local time for fractional Brownian motion, *Ann. Probab.* 2005(33) 948-983.

[7] D. Nualart, G. Peccati, Central limit theorems for sequences of multiple stochastic integrals, *Ann. Probab.* 2005(33) 177-193.

[8] D. Nualart, S. Ortiz-Latorre, Central limit theorems for multiple stochastic integrals and Malliavin calculus, *Stoch. Proc. Appl.* 2008(118) 614-628.

[9] G. Peccati, C.A. Tudor, Gaussian limits for vector-valued multiple stochastic integrals, in: M. Emery, M. Ledoux and M. Yor (eds), *Seminaire de Probabilites XXXVIII*, Lecture Notes in Math 1857, pp. 247-262. Springer-Verlag, Berlin 2005.