Ambit Processes

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Synopsis

- Ambit processes
- Turbulence
- $\mathcal{BSS}$ processes
- Multipower Variations for $\mathcal{BSS}$ processes
- Realised Variation Ratio (RVR)
- $\mathcal{BSS}$ and Wold-Karhunen
- Energy Markets
- Further points

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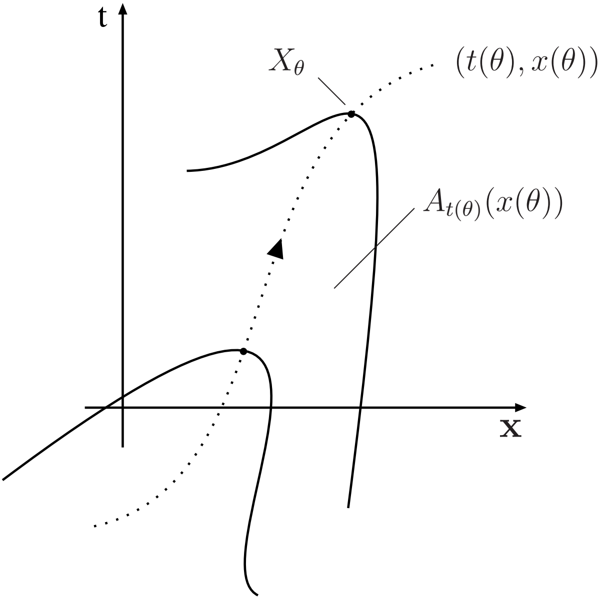

Figure: Ambit processes

Ambit fields

$$\begin{aligned} Y_t(x) &= \mu + \int_{A_t(x)} g(\xi, s; t, x)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) \\ &\quad + \int_{D_t(x)} q(\xi, s; t, x)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s. \end{aligned}$$

Here $A_t(x)$, and $D_t(x)$ are termed *ambit sets*, g and q are deterministic (matrix) functions, $\sigma \geq 0$ is a stochastic field referred to as the *intermittency* or *volatility*, and L is a Lévy basis. Integration in the sense of random measures, as defined by Rajput and Rosinski (1989)

Ambit processes

$$X_\theta = Y_{t(\theta)}(x(\theta)).$$

General form

Volatility Modulated Volterra Ambit Processes (VMVAP)

Stationary regimes

To model stationary fields and processes the damping functions g and q are chosen to have the form

$$g(\xi, s; t, x) = g(x - \xi, t - s)$$

$$q(\xi, s; t, x) = q(x - \xi, t - s)$$

and the ambit sets are taken to be *homogeneous* and nonanticipative, i.e. $A_t(x)$ is of the form $A_t(x) = A + (x, t)$ where A only involves negative time coordinates, and similarly for $D_t(x)$.

Stylized features of isotropic, homogeneous and steady turbulent flows away from boundaries

- *Inertial range scaling relations*
  - scaling of structure functions (K41, K62)
  - scaling exponents
  - role of skewness
  - scaling of the integrated energy dissipation (K62, RQV)
  - scaling of energy dissipation correlators

Second order structure function $S_2(\delta)$

$$S_2(\delta) = const \cdot \mathrm{E}\{\mathrm{RQV}(\delta)\}$$

$$\mathrm{RQV}(\delta) = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(\Delta_j^n Y\right)^2$$

where

$$\Delta_j^n Y = Y_{j\delta} - Y_{(j-1)\delta}$$

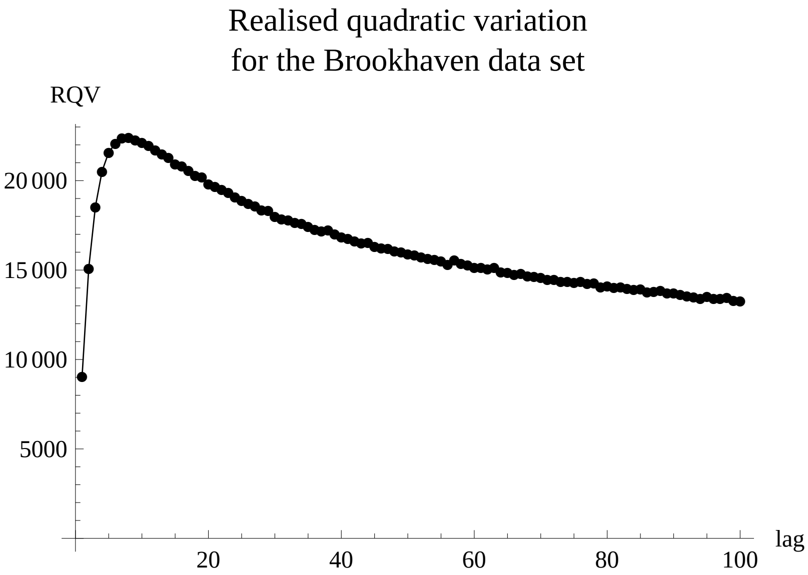

Figure: S_2 Brookhaven data

Log–log–plot of the second order structure function for the Brookhaven data set. Red line has slope 2/3.

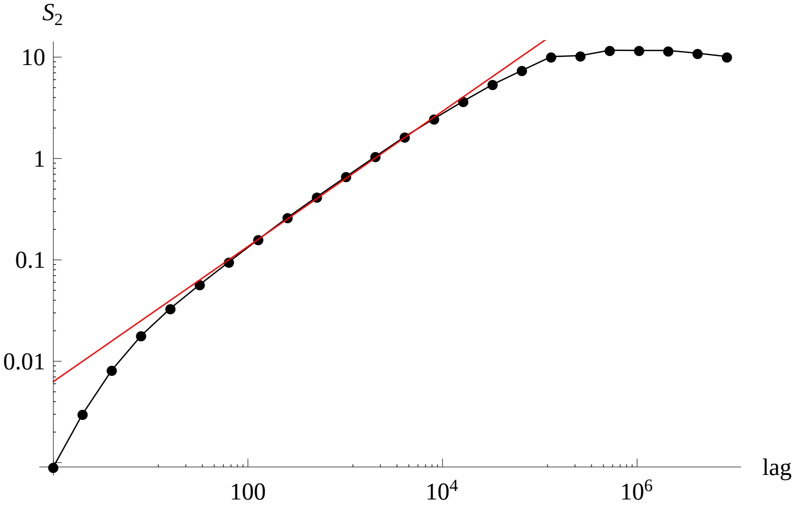

Figure: S_2 Brookhaven data

- *Dissipation scales*:
    - deviations from inertial range scaling
    - small scale diffusion
- *Variation measures*:
    - delta RVR
    - diamond RVR

- *Densities of velocity increments*:
  - evolution across scales
  - heavy tails
  - NIG representation
  - universality

Velocity field

$$
\begin{aligned}
Y_t(x) &= \mu + \int_{A_t(x)} g(t-s, \xi - x)\, \sigma_s(\xi)\, W(\mathrm{d}\xi, \mathrm{d}s) \\
&\quad + \int_{D_t(x)} q(t-s, \xi - x)\, \sigma_s^2(\xi)\, \mathrm{d}\xi \mathrm{d}s.
\end{aligned}
$$

Energy dissipation field:

$$
\sigma_t^2(x) = \int_{C_t(x)} h(t-s, \xi - x)\, L(\mathrm{d}\xi, \mathrm{d}s)
$$

where L is a nonnegative Lévy basis.

Intermittency field:

Model $\log \sigma_t^2(x)$ in the same manner. Important for scaling analysis.

Turbulence background

Example of Ambit sets (for modelling intermittency fields)

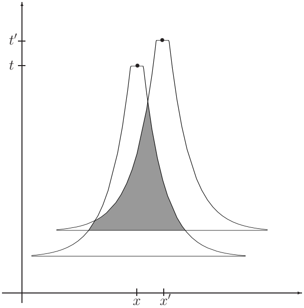

Figure: Ambit sets

Taylor Frozen Field Hypothesis

Most extensive data sets on turbulent velocities only provide the time series of the main component of the velocity vector (i.e. the component in the main direction of the fluid flow) at a single location in space.

The turbulence modelling framework then particularise to the class of $\mathcal{BSS}$ models (Brownian semistationary processes). We discuss this class next, returning at the end to some discussion of the further intriguing issues that arise when addressing tempo-spatial settings.

Brownian semistationary ($\mathcal{BSS}$) processes:

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s) a_s \mathrm{d}s$$

where W is Brownian measure on $\mathbb{R}$, σ and a are cadlag processes and g and q are deterministic continuous memory function on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$.

When σ and a are stationary, as will be assumed throughout this talk, then so is Y.

It is sometimes convenient to indicate the formula for Y as

$$Y = g * \sigma \bullet W + q * a \bullet leb.$$

We consider the $\mathcal{BSS}$ processes to be the natural analogue, in stationarity related settings, of the class $\mathcal{BSM}$ of Brownian semimartingales.

$$Y_t = \int_0^t \sigma_s \mathrm{d}W_s + \int_0^t a_s \mathrm{d}s.$$

- The $\mathcal{BSS}$ processes are not in general semimartingales

Example Suppose $Y = g * \sigma \bullet W$ with $g(t) = t^{\nu-1}e^{-\lambda t}$.

$\frac{1}{2} < \nu < 1$	nonSM
$\nu = 1$	SM
$1 < \nu < \frac{3}{2}$	nonSM

A key object of interest is the *integrated variance* (IV)

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 \mathrm{d}s$$

for any $t \in \mathbb{R}$.

We shall discuss to what extent *realised multipower (in particular quadratic) variations* of Y can be used to estimate σ_t^{2+}.

- Note that the relevant question here is whether a suitably normalised version of the realised quadratic variation, and not necessarily the realised quadratic variation itself, converges in probability.

Multipower Variations (MPV) Let X be a stochastic process in continuous time, observed over the interval $[0, t]$ at time points $0, \delta, 2\delta, \ldots$, where $\delta = \frac{1}{n}$ for some positive integer n.

Realised multipower Δ-variations

A realised multipower Δ-variation of a stochastic process X is an object of the type

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^{n} X|^{p_j}$$

where $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ and $p_1, \ldots, p_k \geq 0$.

Realised multipower $\Diamond$-variations

A realised multipower $\Diamond$-variation of a stochastic process X is an object of the type

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Diamond_{i+j-1}^{n} X|^{p_j}$$

where $\Diamond_i^n X = X_{i\delta} - 2X_{(i-1)\delta} + X_{(i-2)\delta}$ and $p_1, \ldots, p_k \geq 0$.

Multipower variation for $\mathcal{BSS}$ processes

Now consider a $\mathcal{BSS}$ process

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s.$$

Let G be the *Gaussian core* of Y, i.e.

$$G_t = \int_{-\infty}^{t} g(t-s) W(\mathrm{d}s)$$

and let $\mathcal{G}$ be the σ-algebra generated by G.

Key quantities

Define r_n^{Δ} as the autocorrelation function of the Δ-increments of G, i.e.

$$r_n^{\Delta}(j) = \text{cov}\left(\frac{\Delta_1^n G}{\tau_n^{\Delta}}, \frac{\Delta_{1+j}^n G}{\tau_n^{\Delta}}\right)$$

and $r_n^{\Diamond}$ as the autocorrelation function of the $\Diamond$-increments of G, i.e.

$$r_n^{\Diamond}(j) = \text{cov}\left(\frac{\Diamond_1^n G}{\tau_n^{\Diamond}}, \frac{\Diamond_{1+j}^n G}{\tau_n^{\Diamond}}\right)$$

where

$$\left(\tau_n^{\Delta}\right)^2 = \mathrm{E}\left\{|\Delta_1^n G|^2\right\} \textit{and} \left(\tau_n^{\Diamond}\right)^2 = \mathrm{E}\left\{|\Diamond_1^n G|^2\right\}.$$

Let π_δ^Δ be the measure on $\mathbb{R}_+$ defined by

$$\pi_\delta^\Delta(A) = \frac{\int_A (g(x-\delta) - g(x))^2 \mathrm{d}x}{\int_0^\infty (g(x-\delta) - g(x))^2 \mathrm{d}x}.$$

Note that π_δ^Δ is a probability measure on $\mathbb{R}_+$, and set $\bar{\pi}_\delta^\Delta(x) = \pi_\delta^\Delta(\{y : \ y > x\})$.

This measure $\bar{\pi}_\delta$ has a crucial influence on the asymptotic behaviour of the realised multipower Δ-variations of Y.

Let $\pi_\delta^\lozenge$ be the measure on $\mathbb{R}_+$ defined by

$$\pi_\delta^\lozenge(A) = \frac{\int_A (g(x-2\delta) - 2g(x-\delta) + g(x))^2 \mathrm{d}x}{\int_0^\infty (g(x-2\delta) - 2g(x-\delta) + g(x))^2 \mathrm{d}x}.$$

Note that $\pi_\delta^\lozenge$ is a probability measure on $\mathbb{R}_+$, and set $\bar{\pi}_\delta^\lozenge(x) = \pi_\delta^\lozenge(\{y : \ y > x\})$.

This measure $\bar{\pi}_\delta^\lozenge$ has a crucial influence on the asymptotic behaviour of the realised multipower $\lozenge$-variations of Y.

We are interested in the probabilistic limit behaviour of the *normalised* realised multipower Δ-variations

$$\bar{V}_{\Delta}(Y, p_1, \ldots, p_k)_t^n = \frac{1}{n\,(\tau_n^{\Delta})^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j}$$

and of the *normalised* realised multipower $\Diamond$-variations

$$\bar{V}_{\Diamond}(Y, p_1, \ldots, p_k)_t^n = \frac{1}{n\left(\tau_n^{\Diamond}\right)^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Diamond_{i+j-1}^n Y|^{p_j}$$

A basic result for determining the limit behaviour is a *CLT for Gaussian triangular arrays* (this was established using Malliavin calculus).

Example of results obtained (under regularity conditions):

Joint central limit theorem for a family $(\bar{V}(Y, p_1^j, \ldots, p_k^j)_t^n)_{1 \le j \le d}$ *of multipower variations* :

$$\sqrt{n}\Big(\bar{V}_\Delta(Y, p_1^j, \ldots, p_k^j)_t^n - \rho^{(n)}_{p_1^j, \ldots, p_k^j} \int_0^t |\sigma_s|^{p_+^j} \mathrm{d}s\Big)_{1 \le j \le d} \xrightarrow{\mathcal{G}-st} \int_0^t Z_s^{1/2} \mathrm{d}B_s$$

where B is a d-dimensional Brownian motion that is independent of Y, and Z is a $d \times d$-dimensional process

$$Z_s^{ij} = \beta_{ij} |\sigma_s|^{p_+^i + p_+^j} , \qquad 1 \le i, j \le d .$$

Feasible inference

Key example:

$$g(t) = t^{\nu-1}e^{-\lambda t}$$

for $0 < t < \infty$ and $\lambda > 0$ and with $\nu > \frac{1}{2}$.

This allows many explicit calculations and is a useful initial choice for the turbulence modelling.

The upshot of the calculations is that:

- the Δ-LLN holds for $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and that the Δ-CLT is valid provided $\nu \in \left(\frac{1}{2}, \frac{5}{4}\right)$.
- the $\Diamond$-LLN holds for $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and that the $\Diamond$-CLT is valid provided $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$.

RVR

Let us consider the realised Δ-variation ratio (RVR) defined as

$$RVR_t^{\Delta}(\delta) = \frac{\frac{\pi}{2} V^{\Delta}(Y,1,1)_t^n}{V^{\Delta}(Y,2,0)_t^n}.$$

Here $V^{\Delta}(Y,1,1)_t^n$ is the *bipower variation* of Y

Note The RVR_t^{Δ} is calculable without a model

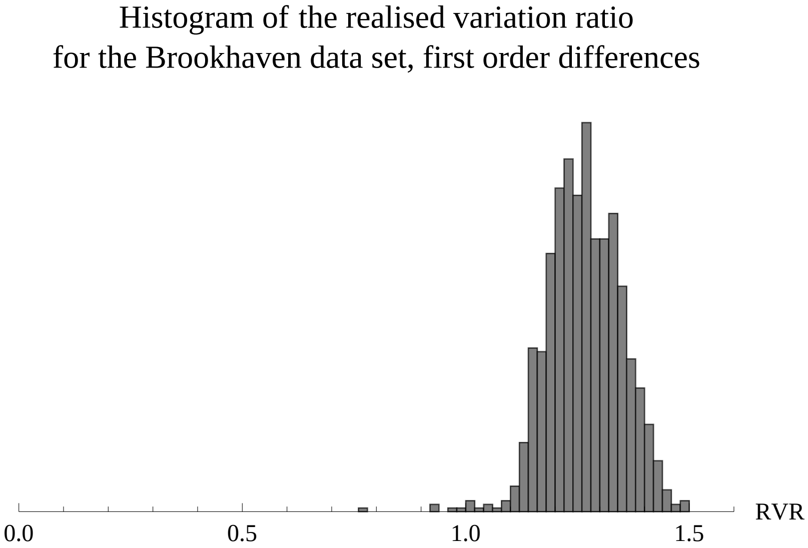

Figure: RVR Brookhaven data

As a consequence of the law of large numbers we obtain the following probability limit result for the realised variatio ratio:

$$RVR_t^{\Delta}(\delta) \xrightarrow{ucp} \psi(\rho(1))$$

where

$$\psi(\rho) = \sqrt{1-\rho^2} + \rho \arcsin \rho$$

which equals $\frac{\pi}{2}\mathrm{E}\{|UV|\}$ where U and V are two standard normal variables with correlation ρ.

Roles of ambit sets and Pii measures

Example Suppose that

$$g(t) = e^{-\lambda t} 1_{(0,l)}(t)$$

with $\lambda > 0$. This is a non-semimartingale case, and it can be shown that

$$\bar{V}(Y,2)_t^n \stackrel{P}{\longrightarrow} \left(1 + e^{-2\lambda l}\right)^{-1} \sigma_t^{2+} - \left(1 + e^{2\lambda l}\right)^{-1} (\sigma_{t-l}^{2+} - \sigma_{-l}^{2+}) \neq \sigma_t^{2+}.$$

Thus, in particular, we do not have $\bar{V}(Y,2)_t^n \stackrel{P}{\longrightarrow} \sigma_t^{2+}$.

Pick up of information on intermittency

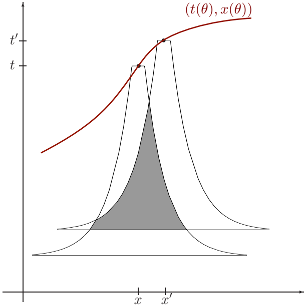

Figure: increments

ambit sets bounded; boundary curve regular; g 'regular'.

- Bounded versus unbounded $A \cap \{g > 0\}$

Pii measure

$$\pi_\delta(B) = \frac{\int_B (g(x-\delta) - g(x))^2 \mathrm{d}x}{\int_{\mathbb{R}^d} (g(x-\delta) - g(x))^2 \mathrm{d}x}$$

Under regularity conditions, for $\delta \longrightarrow 0$

$$\bar{V}_\Delta | \sigma \xrightarrow{p} \int_{\mathbb{R}^2} \int_0^t \sigma^2_{x(\theta)-u} \left(t(\theta) - v \right) m \left(\mathrm{d}u \mathrm{d}v \right).$$

Wold-Karhunen representations

As a modelling framework for continuous time stationary processes the $\mathcal{BSS}$ specification is quite general. In fact, the continuous time Wold-Karhunen decomposition says that any second order stationary stochastic process, possibly complex valued, of mean 0 and continuous in quadratic mean can be represented as

$$Z_t = \int_{-\infty}^{t} \phi\left(t-s\right) \mathrm{d}\Xi_s + V_t. \tag{1}$$

where

- the deterministic function ϕ is an, in general complex, deterministic square integrable function
- the process Ξ has orthogonal increments with $E\left\{\left|\mathrm{d}\Xi_t\right|^2\right\} = \varpi dt$ for some constant $\varpi > 0$
- the process V is nonregular (i.e. its future values can be predicted, in the L^2 sense, by linear operations on past values without error).

Stationary ambit processes: $X_t = Y_t(x(t))$ where

$$\begin{aligned} Y_t(x) &= \mu + \int_{A_t(x)} g(t-s, \xi - x)\, \sigma_s(\xi)\, W(\mathrm{d}\xi, \mathrm{d}s) \\ &\quad + \int_{D_t(x)} q(t-s, \xi - x)\, \sigma_s^2(\xi)\, \mathrm{d}\xi \mathrm{d}s. \end{aligned}$$

So there exists a ϕ such that

$$X_t = \int_{-\infty}^{t} \phi(t-s)\, \mathrm{d}\Xi_s + V_t.$$

What is ϕ? and Ξ?

Example *Fractional Gaussian Ornstein-Uhlenbeck process*

$$\begin{aligned} Y_t &= \int_{-\infty}^{t} e^{-\lambda(t-s)} \mathrm{d}B_s^H \\ &= \int_{-\infty}^{t} \phi^H(t-s)\, \mathrm{d}B_s \end{aligned}$$

where

$$\phi^H(t-s) = ?$$

Energy Markets

$\mathcal{LSS}$ processes

Stylised facts:

- Samuelson effect
- High correlation between neighbouring contracts near maturity

Maturity at time $T = t_0 + T_0$. Time to maturity u.

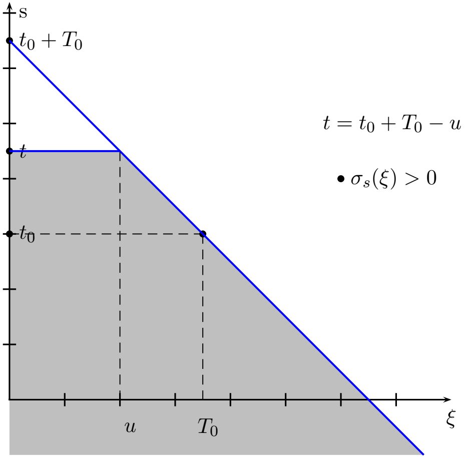

Figure: spot/forward

To what extent can one create a stochastic calculus for (stationary) ambit processes?

"Stochastic Differentials"?

"Ito Algebra"?

.....

$$
\begin{aligned}
Y_t &= \int_{t-l}^{t} e^{-\lambda(t-s)} \sigma_s W(\mathrm{d}s) \\
&= X_t - e^{-\lambda l} X_{t-l}
\end{aligned}
$$

where

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} \sigma_s W(\mathrm{d}s).$$

$$
\begin{aligned}
\text{"}\mathrm{d}Y_t\text{"} &= \mathrm{d}X_t - e^{-\lambda l}\mathrm{d}X_{t-l} \\
&= \sigma_t W(\mathrm{d}t) - \lambda X_t \mathrm{d}t - e^{-\lambda l}\sigma_{t-l} W(\mathrm{d}(t-l)) + \lambda e^{-\lambda l} X_{t-l}\mathrm{d}t
\end{aligned}
$$

so

$$
\begin{aligned}
(\text{"}\mathrm{d}Y_t\text{"})^2 &= \left(\sigma_t^2 + e^{-2\lambda l}\sigma_{t-l}^2\right)\mathrm{d}t \\
&= \text{"}\mathrm{d}[Y]_t\text{"}.
\end{aligned}
$$

Alignment

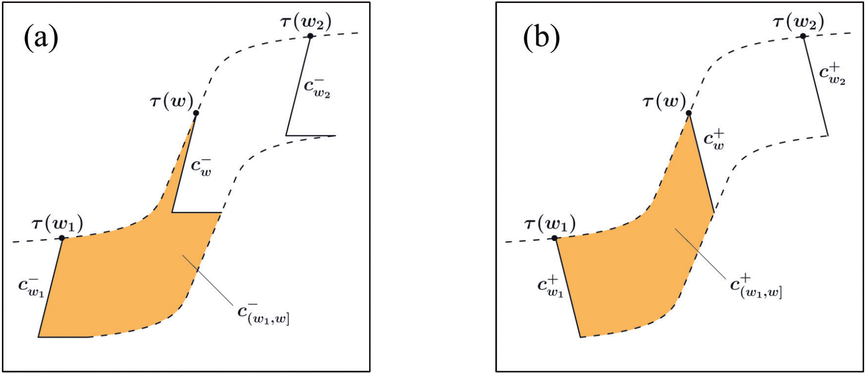

Figure: Aligned

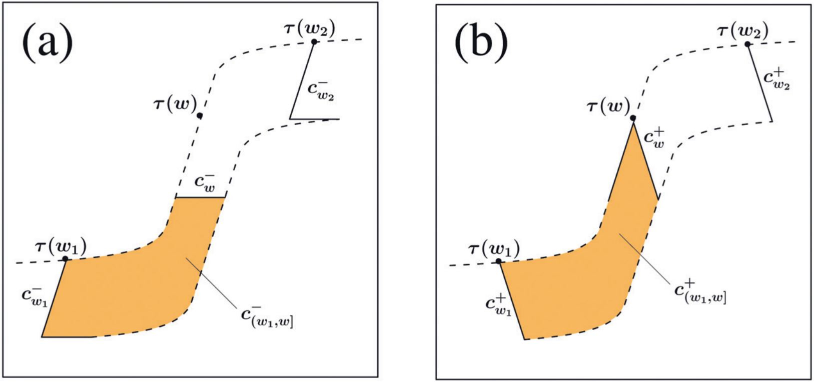

Figure: Nonaligned