Quasi Ornstein-Uhlenbeck Processes

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The Ornstein-Uhlenbeck process

- Suppose we are given a free particle immersed in a liquid with velocity v_t and mass m. Then the physical description of the particles motion is described by the *Langevin equation* (see Langevin (1908))

$$m\frac{dv_t}{dt} = -\zeta v_t + \dot{N}_t.$$

where ζ is the friction constant and $\dot{N}_t$ is a fluctuation force.

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- Uhlenbeck and Ornstein (1930) imposed the assumption that $\dot{N}$ is a white noise, i.e., the formal derivative of a Wiener process N. Hence they arrived with the equation:

$$\mathrm{d}v_t = -(\zeta/m)v_t\,\mathrm{d}t + (1/m)\,\mathrm{d}N_t$$

  that is,

$$v_t = v_0 - \frac{\zeta}{m}\int_0^t v_s\,\mathrm{d}s + \frac{1}{m}N_t, \qquad t \in \mathbb{R},$$

  which today is known as the Ornstein-Uhlenbeck process.

- In this talk we will be concerned with the situation where the noise N has memory, i.e., dependent increments.

quasi Ornstein-Uhlenbeck processes (QOUs)

- Let $\lambda > 0$ and $N = (N_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and $N_0 = 0$; that is, $(\omega, t) \mapsto N_t(\omega)$ is measurable and for all $s \in \mathbb{R}$,

$$(N_t - N_0)_{t\in\mathbb{R}} \overset{\mathcal{D}}{=} (N_{t+s} - N_s)_{t\in\mathbb{R}}.$$

- By a quasi Ornstein-Uhlenbeck process (QOU) we mean a stationary solution $X = (X_t)_{t\in\mathbb{R}}$ to the Langevin equation

$$\mathrm{d}X_t = -\lambda X_t \,\mathrm{d}t + \mathrm{d}N_t,$$

that is, X is a stationary process such that for all $t, u \in \mathbb{R}$ with $u < t$ we have that

$$X_t - X_u = -\lambda \int_u^t X_s \,\mathrm{d}s + N_t - N_u.$$

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The Lévy case

Recall the following classical result:

Theorem (Wolfe (1982) and Sato and Yamazato (1983))

Assume that N is a Lévy process. Then, there exists a QOU process X driven by N if and only if $\mathrm{E}[\log^+|N_t|] < \infty$ *for all* $t \in \mathbb{R}$. *In this case the solution is unique in law and given by*

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)}\,\mathrm{d}N_s, \qquad t \in \mathbb{R}. \tag{1}$$

- Note that, a random variable is selfdecomposable if and only if it is of the form (1).
- When N has dependent increments, X is, in general, not Markovian.

The linear fractional stable motion

- The *linear fractional stable motion* (LFSM) of indexes $\alpha \in (0,2]$ and $H \in (0,1)$ is a important example of a N; here

$$N_t = \int_{-\infty}^{t} \left[(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right] \mathrm{d}Z_s, \qquad t \in \mathbb{R}$$

  and $Z = (Z_t)_{t\in\mathbb{R}}$ is a symmetric α-stable Lévy process.
- The QOU process driven by a fractional Brownian motion is often called a fractional Ornstein-Uhlenbeck process; see Cheriditio, Kawaguchi and Maejima (2003).
- For the results on the case where N is a LFSM with $\alpha \in (1,2)$, see Maejima and Yamamoto (2003).

Existence and uniqueness of QOUs

A stochastic process $Z = (Z_t)_{t\in\mathbb{R}}$ is said to have finite p-moments if $\mathrm{E}[|Z_t|^p] < \infty$ for all $t \in \mathbb{R}$.

Theorem

Assume that N has finite first-moments. Then there exists a unique in law QOU process X driven by N, and it is given by

$$X_t = N_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \, ds, \qquad t \in \mathbb{R}.$$

Furthermore, if N has finite p-moments for some $p \geq 1$, then X has finite p-moments and is continuous in L^p.

- Note that when N is a LFSM with indexes $\alpha \in (1, 2]$ and $H \in (0, 1)$, N has finite first-moments and hence by the above theorem there exists an unique in law QOU process X driven by N.
- When $H \in (0, 1/\alpha)$ Maejima and Yamamoto (2003) conjectured that there does not exists a QOU process driven by N, due to the fact that the sample paths of N are unbounded on each non-empty interval with probability one.

Continuity result

Lemma

Let $p \geq 0$ and assume that N has finite p-moments. Then, N is continuous in L^p and when $p \geq 1$ there exists $\alpha, \beta \in \mathbb{R}_+$ such that $\|N_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$.

1. The proof relies on an application of the Steinhaus lemma borrowed from Surgailis et al. (1998) together with an extension of a result by Cohn (1972).
2. In fact, we show the above result not only in $L^p(\Omega, \mathcal{F}, P)$, but in all modular spaces $L^\phi(E, \mathcal{E}, \mu)$ where $(E, \mathcal{E}, \mu)$ is a σ-finite measure space and $\phi\colon \mathbb{R} \to \mathbb{R}_+$ is a symmetric continuous function, which is increasing on $\mathbb{R}_+$ and $\phi(0) = 0$. Note that for $\phi(x) = |x|^p$ we have $L^\phi(E, \mathcal{E}, \mu) = L^p(E, \mathcal{E}, \mu)$.

Proof of the existence

The existence of the pathwise Lebesgue integral $\int_{-\infty}^{t} e^{\lambda s} N_s \, \mathrm{d}s$ follows from the above lemma since

$$\mathrm{E}\Big[\int_{-\infty}^{t} e^{\lambda s} |N_s| \, \mathrm{d}s\Big] \leq \int_{-\infty}^{t} e^{\lambda s} \mathrm{E}[|N_s|] \, \mathrm{d}s \leq \int_{-\infty}^{t} e^{\lambda s} (\alpha + \beta|s|) \, \mathrm{d}s < \infty.$$

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Let $X = (X_t)_{t \in \mathbb{R}}$ be defined by

$$X_t = N_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} N_s \, \mathrm{d}s, \qquad t \in \mathbb{R}.$$

By use of partial integration it follows that X satisfies $\mathrm{d}X_t = -\lambda X_t \, \mathrm{d}t + \mathrm{d}N_t$.

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By use of partial integration it follows that X satisfies $\mathrm{d}X_t = -\lambda X_t \, \mathrm{d}t + \mathrm{d}N_t$. Moreover by substitution we have that

$$X_t = \lambda \int_{-\infty}^{0} e^{\lambda s} (N_t - N_{t+s}) \, \mathrm{d}s. \tag{2}$$

Using the L^1-continuity of N from the above lemma it follows that the integral (2) is a limit of Riemann sums in L^1, which together with the stationary increments of N implies that X is stationary.

Mean and variance

Assume that N has finite second-moments and let X be the corresponding QOU process. Moreover, let $\mathrm{V}_N(t) = \mathrm{Var}(N_t)$ for $t \in \mathbb{R}$, denote the variance function of N.

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Theorem

$$\mathrm{E}[X_0] = \frac{\mathrm{E}[N_1]}{\lambda} \qquad \textit{and} \qquad \mathrm{Var}(X_0) = \frac{2}{\lambda}\int_0^\infty e^{-\lambda s}\mathrm{V}_N(s)\,\mathrm{d}s.$$

For example when $N_t = \mu t + \sigma B_t^H$ and B^H is a fBm of index $H \in (0,1)$, we have $\mathrm{V}_N(t) = t^{2H}$ for $t > 0$ and hence

$$\mathrm{E}[X_0] = \frac{\mu}{\lambda} \qquad \text{and} \qquad \mathrm{Var}(X_0) = \frac{\sigma^2\Gamma(1+2H)}{2\lambda^{2H}}.$$

Asymptotic behavior of the autocovariance function

We will write $f(t) \sim g(t)$ for $t \to 0$ (or ∞), when $f(t)/g(t) \to 1$ for $t \to 0$ (or ∞). Let

$$\mathrm{R}_X(t) = \mathrm{Cov}(X_t, X_0) \qquad \text{and} \qquad \overline{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2].$$

Theorem

Assume that N has finite second-moments and let X be the QOU process driven by N.

- *Assume for* $t \to \infty$ *that* $\mathrm{V}_N''(t) = O(e^{(\lambda/2)t})$ *and* $e^{-\lambda t} = o(\mathrm{V}_N''(t))$ *and* $\mathrm{V}_N'''(t) = o(\mathrm{V}_N''(t))$*. Then for* $t \to \infty$ *we have*

$$\mathrm{R}_X(t) \sim \frac{1}{2\lambda^2}\mathrm{V}_N''(t).$$

- *Assume that for* $t \to 0$ *we have* $t^2 = o(\mathrm{V}_N(t))$*. Then for* $t \to 0$ *we have* $\overline{\mathrm{R}}_X(t) \sim \frac{1}{2}\mathrm{V}_N(t)$.

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- Recall that a stationary process $Z = (Z_t)_{t\in\mathbb{R}}$ is said to have *long range dependence of order* $\alpha \in (0,1)$ if its autocovariance function $\mathrm{R}_Z(t)$ is regularly varying of index $-\alpha$ for $t \to \infty$.
- Thus, long range dependence of a QOU process X is the same as that $\mathrm{V}_N''(t)$ is regularly varying of exponent $\beta \in (-1, 0)$ for $t \to \infty$.

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When N is a fBm of index $H \in (0,1)$ we have $\mathrm{V}_N(t) = t^{2H}$ and hence $\mathrm{V}_N''(t) = 2H(2H-1)t^{2H-2}$ for $t > 0$, which shows that for $H \neq 1/2$ and $t \to \infty$ we have

$$\mathrm{R}_X(t) \sim \left(\frac{H(2H-1)}{\lambda^2}\right)t^{2H-2}. \tag{3}$$

The asymptotic behavior (3) in the case of a fBm is also obtained in Cheridito, Kawaguchi and Maejima (2003). Recall that for $H = 1/2$, $\mathrm{R}_X(t) = e^{-\lambda t}/(2\lambda)$.

Moving averages

Let us consider the case where $N = (N_t)_{t\in\mathbb{R}}$ is a pseudo moving average (PMA) of the form

$$N_t = \int_{\mathbb{R}} \big[f(t-s) - f(-s)\big]\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where $Z = (Z_t)_{t\in\mathbb{R}}$ is a centered Lévy process and $f\colon \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\big|x(f(t-s) - f(-s))\big|^2 \wedge \big|x(f(t-s) - f(-s))\big|\Big)\,\nu(\mathrm{d}x)\,\mathrm{d}s < \infty.$$

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From a result by Cohn (1972) we may choose a measurable modification of N.

Theorem

There exists a unique in law QOU process driven by N, and it is a moving average of the form

$$X_t = \int_{\mathbb{R}} \psi_f(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where

$$\psi_f(t) = \Big(f(t) - \lambda e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s} f(s)\,\mathrm{d}s\Big), \qquad t \in \mathbb{R}.$$

Examples include the case where N is the LFSM with $\alpha \in (1,2]$ and $H \in (0,1)$.

Stochastic Fubini

Let Λ be a centered ID random measure on $(S, \mathcal{S})$ where S is a non-empty space and $\mathcal{S}$ is a σ-finite δ-ring on S. Let T be a separable and complete metric space, μ be a σ-finite measure on T and $f : T \times S \to \mathbb{R}$ be a measurable function satisfying

$$\int_T \|f(t,\cdot)\|_\phi \, \mu(\mathrm{d}t) < \infty,$$

where for $y \in \mathbb{R}$ and $s \in S$ we have

$$\phi(y,s) = y^2\sigma^2(s) + \int_{\mathbb{R}} \left(|uy|^2 1_{|uy|\le 1} + (2|uy|-1)1_{|uy|>1}\right) \nu(\mathrm{d}u, s),$$

and $\|\cdot\|_\phi$ is the corresponding Musielak-Orlicz norm on $L^\phi(S, \sigma(\mathcal{S}), m)$.

Theorem (Stochastic Fubini)

All of the below integrals exist and we have

$$\int_S \left(\int_T f(t,s)\,\mu(\mathrm{d}t)\right)\Lambda(\mathrm{d}s) = \int_T \left(\int_S f(t,s)\,\Lambda(\mathrm{d}s)\right)\mu(\mathrm{d}t).$$

The above theorem relies on an inequality by Marcus and Rosiński (2003).

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Asymptotic behavior of the autocovariance function

Consider a moving average $X = (X_t)_{t\in\mathbb{R}}$ of the form

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}. \tag{4}$$

Proposition

Let X be given by (4) and assume that $\psi(t) \sim ct^{\alpha}$ for $t \to \infty$.

- *For $\alpha \in (-1, -\frac{1}{2})$ we have for $t \to \infty$ that $\mathrm{R}_X(t) \sim (c^2 k_\alpha) t^{2\alpha+1}$.*
- *For $\alpha \in (-\infty, -1)$ we have for $t \to \infty$ that $\mathrm{R}_X(t) \sim (c \int_0^\infty \psi(s)\,\mathrm{d}s) t^{\alpha}$, provided $\int_0^\infty \psi(s)\,\mathrm{d}s \neq 0$.*

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Now let N be a PMA of the form $N_t = \int_{-\infty}^{t} (f(t-s) - f(-s))\,\mathrm{d}Z_s$ and let X be the QOU process driven by N.

Proposition

Let $\alpha \in (-1, -\frac{1}{2})$ and assume that $f \in C^1((\beta, \infty); \mathbb{R})$ with $f'(t) \sim ct^{\alpha}$. Then for $t \to \infty$ we have $\mathrm{R}_X(t) \sim (\frac{c^2 k_\alpha}{\lambda^2}) t^{2\alpha+1}$.

Stability of the autocovariance function

For simplicity let us consider the case where N if a FBM of index $H \in (0,1) \setminus \{1/2\}$. Let X be the QOU process driven by N, that is,

$$X_t = \int_{-\infty}^t \psi_H(t-s)\,\mathrm{d}Z_s, \qquad \psi_H(t) = c_H\big(t^{H-1/2} - \lambda e^{-\lambda t}\int_0^t e^{\lambda s} s^{H-1/2}\,\mathrm{d}s\big).$$

For each bounded function $f\colon \mathbb{R} \to \mathbb{R}$ with compact support let

$$Y_t^f = \int_{-\infty}^t f(t-s)\,\mathrm{d}Z_s \qquad \text{and} \qquad X_t^f = X_t + Y_t^f.$$

Note that $\mathrm{R}_{Y^f}(t) = 0$ for t large.

Corollary

For some $c_1, c_2, c_3 \neq 0$ we have

- *For $H \in (0, \frac{1}{2})$ and if $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$, then for $t \to \infty$ we have*

$$\mathrm{R}_{X^f}(t) \sim c_2 \mathrm{R}_X(t) t^{1/2-H} \sim c_1 t^{H-3/2}.$$

- *For $H \in (\frac{1}{2}, 1)$, then for $t \to \infty$ we have*

$$\mathrm{R}_{X^f}(t) \sim \mathrm{R}_X(t) \sim c_3 t^{2H-2}.$$

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