On the Infinite Divisibility of Power Semicircle Distributions

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Outline of the Talk

I. **Semicircle or Wigner Distribution**
   1. Density and Moments
   2. Importance in Mathematics

II. **Power Semicircle Distributions**
   1. Representation of the Classical Gaussian Distribution
   2. Poincaré´s Theorem
   3. Recursive Representations
   4. Moments

III. **Review of Steen´s talk:**
   1. Transforms of Measures
   2. Cumulant transforms
   3. Convolutions corresponding to classes of independence.

IV. **Kurtosis and Infinite Divisibility**
   1. Kurtosis corresponding to the five classes of independence
   2. Necessary condition for ID based on kurtosis
   3. Applications to Power Semicircle Distributions

V. **Conjectures and Open Problems**

- **Semicircle or Wigner distribution** on $(-\sigma, \sigma)$, $\sigma > 0$, has density

$$f(x\,;\sigma) = \frac{2}{\pi\sigma^2}\sqrt{\sigma^2 - x^2}1_{(-\sigma,\sigma)}(x).$$

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Definition and Basic Properties

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- Semicircle distribution plays an important role in several fields of mathematics and its applications.

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- Moreover

$$\mathrm{w}_t(\mathrm{d}x) = \frac{1}{2t\pi}\sqrt{4t - x^2}1_{[-\sqrt{4t},\sqrt{4t}]}\mathrm{d}x, \, t > 0$$

is the family of spectral distributions of the so called **free Brownian motion.**

II. Power Semicircle Distributions

- Kingman (Acta Math. (1963)): **Power semicircle law** $(PS(\theta, \sigma))$: $\theta \geq -3/2$, $\sigma > 0$

$$f_\theta(x\,;\sigma) = c_{\theta,\sigma}\left(\frac{2}{\pi\sigma^2}\sqrt{\sigma^2 - x^2}\right)^{2\theta+1} \qquad -\sigma < x < \sigma$$

where

$$c_{\theta,\sigma}^{-1} = \sqrt{\pi}\sigma^{-2\theta}\frac{\Gamma(\theta + 3/2)}{\Gamma(\theta + 2)}.$$

- Bounded supported distributions.
- θ shape parameter, σ range parameter.
- When $d = 2(\theta + 2)$ is integer, $PS(\theta, \sigma)$ is the distribution of one-dimensional marginals of uniform measure on a sphere of radius $\sqrt{d}$ in $\mathbb{R}^d$

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5. *Poincaré´s theorem*: $(\theta \to \infty)$

$$f_\theta(x; \sqrt{(\theta+2)/2}\sigma) \to \frac{1}{\sqrt{2\pi}\sigma} \exp(-x^2/(2\sigma^2)).$$

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6. Modelling aspect: finite range distribution and Poincaré´s theorem.

- $G(\alpha, \beta)$ gamma distribution

Theorem

$\theta > -3/2$, $Y_{\theta+2} \sim G(\theta+2, 2)$ independent of $S_\theta \sim PS(\theta, 1)$. Then

$$Z = \sqrt{Y_{\theta+2}} S_\theta \sim N(0,1)$$

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Interested fact related to classical infinite divisibility

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Theorem

$Y_\lambda \sim G(\lambda, 2)$ *independent of* $S_\theta \sim PS(\theta, 1)$. *Then*

$$X = \sqrt{Y_\lambda} S_\theta$$

is infinitely divisible in classical sense if and only if $\lambda = \theta + 2$ *in which case* X *has the Gaussian distribution.*

- For $\theta > -1/2$

$$S_\theta \stackrel{d}{=} U^{1/(2(\theta+1))} S_{\theta-1}$$

where U and $S_{\theta-1}$ are independent $U \sim U(0,1)$ and $S_{\theta-1} \sim PS(\theta-1,1)$

II. Power Semicircle Distributions

Recursion via mixtures

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$$S_{1/2} \stackrel{d}{=} U^{1/3} S_{-1/2}$$

- For $k \geq 1$ integer, $ES_\theta^k = 0$

$$ES_\theta^{2k} = \left(\frac{\sigma}{2}\right)^{2k} C_k(k+1)!\frac{\Gamma(\theta+2)}{\Gamma(\theta+2+k)}$$

- If θ integer

$$ES_\theta^{2k} = \frac{\binom{2k}{k}}{\binom{\theta+k+1}{k}}\left(\frac{\sigma}{2}\right)^{2k}$$

- Standard distribution (zero mean and variance one)

$$\sigma^2 = 2\Gamma(\theta+3)/\Gamma(\theta+2)$$

- θ integer

$$\sigma^2 = 2(\theta+2).$$

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Infinitely Divisible Aspects

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- Question: What about other members of the class of Power Semicircle laws $PS(\theta, 1)$?

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- **Classical kurtosis** of μ

$$Kurt(\mu) = \frac{c_4(\mu)}{(c_2(\mu))^2} = \frac{\widetilde{m}_4(\mu)}{(\widetilde{m}_2(\mu))^2} - 3 \geq -2,$$

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- **Free kurtosis** of μ

$$Kurt^{\boxplus}(\mu) = \frac{k_4(\mu)}{(k_2(\mu))^2} = \frac{\widetilde{m}_4(\mu)}{(\widetilde{m}_2(\mu))^2} - 2 = Kurt(\mu) + 1 \geq -1,$$

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- **Monotone kurtosis** of μ

$$Kurt^{\rhd}(\mu) = \frac{r_4(\mu)}{(r_2(\mu))^2} = \frac{\widetilde{m}_4(\mu)}{(\widetilde{m}_2(\mu))^2} - 1.5 = Kurt(\mu) + 1.5 \geq -\frac{1}{2},$$

$r_2(\mu)$, $r_4(\mu)$ are 2nd and 4th free cumulants (Hasebe-Saigo, 2009).

- **Boolean kurtosis** of μ

$$Kurt^{\uplus}(\mu) = \frac{h_4(\mu)}{(h_2(\mu))^2} = \frac{\widetilde{m}_4(\mu)}{(\widetilde{m}_2(\mu))^2} - 1 = Kurt(\mu) + 2 \geq 0,$$

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$h_2(\mu)$, $h_4(\mu)$ are second and fourth Boolean cumulants (Speicher-Woroudi, 1997).

- In general, if $\circ$ is any of the convolutions associated to the five classes of independence, **Kurtosis with respect to convolution** $\circ$ is

$$Kurt^{\circ}(\mu) = \frac{c_4^{\circ}(\mu)}{(c_2^{\circ}(\mu))^2}$$

Theorem

Let μ be a probability measure on $\mathbb{R}$ with finite fourth moment. If μ is infinitely divisible with respect to $\circ$ then $\mathrm{Kurt}^{\circ}(\mu) \geq 0$.

Theorem

Let μ be a probability measure on $\mathbb{R}$ with finite fourth moment.

a) If μ is ID wrt to classical convolution $\star$, then Kurt$(\mu) \geq 0$.

b) If μ is ID wrt to free convolution $\boxplus$, then Kurt$(\mu) \geq -1$.

c) If μ is ID wrt to monotone convolution $\rhd$, then Kurt$(\mu) \geq -1.5$.

d) If μ is ID wrt to Boolean convolution $\uplus$, then Kurt$(\mu) \geq -2$.

- If S_θ has power semicircle distribution $PS(\theta, 1), \theta \geq -3/2$.

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IV. Kurtosis and Infinite Divisibility

Application to Power Semicircle distribution

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- $PS(\theta, \sigma)$ is not ID in the monotone if $\theta < -1$.
- $PS(\theta, \sigma)$ is not ID in the Boolean sense if $\theta < -3/2$ (in fact is not a distribution).

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- Conjecture 2 has been recently been proved to be true by Belischi, Bozejko, Lehner and Speicher (2009).
- Conjecture 3: The classical Gaussian distribution is the free multiplicative convolution of the semicircle distribution (Lévy Conference in Dresden?)