Riemann-Stieltjes integrals and fractional Brownian motion

Esko Valkeila

Aalto University, School of Science and Engineering, Department of Mathematics and Systems Analysis

Workshop on Ambit processes, Sandbjerg, Monday January 25, 2010, 15.00–15.30

Aalto University

Co-authors and contents

The talk is based on joint work with Ehsan Azmoodeh (Aalto university), Yuliya Mishura (Kiev) and Heikki Tikanmäki (Aalto university).

Azmoodeh and Tikanmäki are grateful to FGSS for support, Mishura acknowledges the support from Suomalainen Tiedeakatemia, and V. acknowledges partial support from the Academy of Finland.

- Motivation

Aalto University

Co-authors and contents

The talk is based on joint work with Ehsan Azmoodeh (Aalto university), Yuliya Mishura (Kiev) and Heikki Tikanmäki (Aalto university).

Azmoodeh and Tikanmäki are grateful to FGSS for support, Mishura acknowledges the support from Suomalainen Tiedeakatemia, and V. acknowledges partial support from the Academy of Finland.

- Motivation
- Stochastic integrals

Co-authors and contents

The talk is based on joint work with Ehsan Azmoodeh (Aalto university), Yuliya Mishura (Kiev) and Heikki Tikanmäki (Aalto university).
Azmoodeh and Tikanmäki are grateful to FGSS for support, Mishura acknowledges the support from Suomalainen Tiedeakatemia, and V. acknowledges partial support from the Academy of Finland.

- Motivation
- Stochastic integrals
- Main results

Co-authors and contents

The talk is based on joint work with Ehsan Azmoodeh (Aalto university), Yuliya Mishura (Kiev) and Heikki Tikanmäki (Aalto university).
Azmoodeh and Tikanmäki are grateful to FGSS for support, Mishura acknowledges the support from Suomalainen Tiedeakatemia, and V. acknowledges partial support from the Academy of Finland.

- Motivation
- Stochastic integrals
- Main results
- Speculations

Aalto University

References

- For fractional Brownian motion we use the book by Mishura: *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, 2008.

References

- For fractional Brownian motion we use the book by Mishura: *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, 2008.
- For the section *Main results* we use the paper Azmoodeh, Mishura, V.: *On hedging European options in geometric fractional Brownian motion market model*, Statistics & Decisions (2010), forthcoming, and the manuscript Azmoodeh, Tikanmäki, V. (2009), where we try to study the hedging of look-back options in geometric fractional Brownian market model.

References

- For fractional Brownian motion we use the book by Mishura: *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, 2008.
- For the section *Main results* we use the paper Azmoodeh, Mishura, V.: *On hedging European options in geometric fractional Brownian motion market model*, Statistics & Decisions (2010), forthcoming, and the manuscript Azmoodeh, Tikanmäki, V. (2009), where we try to study the hedging of look-back options in geometric fractional Brownian market model.
- For the section *Speculations* we use the paper Bender, Sottinen, V. *Pricing by hedging beyond semimartingales*, Finance & Stochastics, 2008..

Motivation

Representation theorem for Brownian functionals

We work with a probability space $(\Omega, F, \mathbf{P})$.
Let W be a standard Brownian motion, and $\mathbb{F}^W$ is the intrinsic filtration of W: $F_t^W = \sigma\{W_s : s \leq t\}$.

Motivation

Representation theorem for Brownian functionals

We work with a probability space $(\Omega, F, \mathbf{P})$.
Let W be a standard Brownian motion, and $\mathbb{F}^W$ is the intrinsic filtration of W: $F_t^W = \sigma\{W_s : s \leq t\}$.
The following representation theorem for square integrable functionals $Y \in L^2(F_T^W)$ is well-known:

$$F = \mathrm{E}\, F + \int_0^T H_s^F \, dW_s;$$

here H^F is a predictable process with the property $\mathrm{E} \int_0^T \left(H_s^F\right)^2 ds < \infty$.

Motivation

Representation theorem for Brownian functionals

We work with a probability space $(\Omega, F, \mathbf{P})$.
Let W be a standard Brownian motion, and $\mathbb{F}^W$ is the intrinsic filtration of W: $F_t^W = \sigma\{W_s : s \leq t\}$.
The following representation theorem for square integrable functionals $Y \in L^2(F_T^W)$ is well-known:

$$F = \mathrm{E}\, F + \int_0^T H_s^F \, dW_s;$$

here H^F is a predictable process with the property $\mathrm{E} \int_0^T \left(H_s^F\right)^2 ds < \infty$.
Let B^H be a fractional Brownian motion: B^H is a continuous centered Gaussian process with covariance

$$\mathrm{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Motivation

Representation theorem for fractional Brownian functionals?

One can show that $F_t^W = F_t^{B^H}$, $0 \leq t \leq T$. Hence, if the random variable $F \in L^2(F_T^W)$, then we automatically have that $F \in L^2(F_T^{B^H})$. In plain English: every Brownian functional is also a fractional Brownian functional.

Motivation

Representation theorem for fractional Brownian functionals?

One can show that $F_t^W = F_t^{B^H}$, $0 \leq t \leq T$. Hence, if the random variable $F \in L^2(F_T^W)$, then we automatically have that $F \in L^2(F_T^{B^H})$. In plain English: every Brownian functional is also a fractional Brownian functional.
It is then natural to ask, if every square integrable fractional Brownian functional has a representation as a stochastic integral with respect to fractional Brownian motion?

Motivation

Representation theorem for fractional Brownian functionals?

One can show that $F_t^W = F_t^{B^H}$, $0 \leq t \leq T$. Hence, if the random variable $F \in L^2(F_T^W)$, then we automatically have that $F \in L^2(F_T^{B^H})$. In plain English: every Brownian functional is also a fractional Brownian functional.
It is then natural to ask, if every square integrable fractional Brownian functional has a representation as a stochastic integral with respect to fractional Brownian motion?
Now we will have some problems:

- Fractional Brownian motion is not a semimartingale, and then the definition of the integrals is not clear at all.
- In contrast to the Brownian case, we already have the first negative result: if $Y \in \overline{\text{span}}\{B_s^H : s \leq t\}$ for $H > \frac{1}{2}$, then Y may fail to have a representation as $Y = \int_0^t f_s dB_s^H$, where f is a deterministic function [Pipiras & Taqqu, Molchan].

Aalto University

Stochastic intgerals

Representation theorem with the help of abstract 'integrals'

Skorohod integrals and Malliavin calculus are tools to export results from Wiener space to other Gaussian spaces. One can follow this approach and prove the following type of result: if $F \in L^2(F_T^{B^H})$, then one has the following representation for F

$$F = \mathrm{E}\, F + \int_0^T H_s^F \delta B_s^H,$$

and this 'integral' is understood as Skorohod 'integral', or as an adjoint of Malliavin derivative.

Stochastic intgerals

Representation theorem with the help of abstract 'integrals'

Skorohod integrals and Malliavin calculus are tools to export results from Wiener space to other Gaussian spaces. One can follow this approach and prove the following type of result: if $F \in L^2(F_T^{B^H})$, then one has the following representation for F

$$F = \operatorname{E} F + \int_0^T H_s^F \delta B_s^H,$$

and this 'integral' is understood as Skorohod 'integral', or as an adjoint of Malliavin derivative.

But we continue to have problems:

- If you want that for $s < t$ the following holds

$$\int_0^t H_u \delta B_u^H = \int_0^s H_u \delta B_u^H + \int_s^t H_u \delta B_u^H,$$

one must use non-anticipative integrands H.

Stochastic integrals

Representation theorem with the help of abstract 'integrals'; Stieltjes integrals

The integration theory with abstract 'integrals' is difficult to interpret in some applications. On the other hand, Stieltjes integrals with respect to fractional Brownian motion, $H > \frac{1}{2}$, can be reasonably interpreted in these applications.

Stochastic integrals

Representation theorem with the help of abstract 'integrals'; Stieltjes integrals

The integration theory with abstract 'integrals' is difficult to interpret in some applications. On the other hand, Stieltjes integrals with respect to fractional Brownian motion, $H > \frac{1}{2}$, can be reasonably interpreted in these applications.

We have now different type of problems:

- Which random variables $Y \in L^2(F_T^{B^H})$ have a representation

$$Y = C + \int_0^T H_s^Y \, dB_s^H,$$

  where the integral is a Riemann-Stieltjes integral.

- Fact: if $Y = F(B_T^H)$ with $F \in C_1(\mathbb{R})$, then

$$Y = F(0) + \int_0^T F_x(B_s^H) dB_s^H.$$

Main results

Convex functions of B_T^H

The next result is by Azmoodeh, Mishura, V.: Assume that F is a convex function with the right derivative F_x^+. Then we have the representation

$$F(B_T^H) = F(0) + \int_0^T F_x^+(B_s^H) dB_s^H. \tag{1}$$

- What is a bit surpising in (1) is the fact that the integral on the right hand side is a Riemann-Stieltjes integral: if one applies the change of variables formula (1) to the function $f(x) = |x|$ one obtains

$$|B_T^H| = \int_0^T \operatorname{sgn}(B_s^H) dB_s^H,$$

  and the process $\operatorname{sgn}(B^H)$ has unbounded variation on compacts.

Main results

Running maximum of a continuous bounded variation function A

The change of variables formula (1) is the same for a continuous bounded variation functions A: if F is convex, then

$$F(A_T) = F(A_0) + \int_0^T F_x^+(A_s)dA_s.$$

- A natural question: to what extend fractional Brownian motion behaves as a continuous function with bounded variation?

Main results

Running maximum of a continuous bounded variation function A

The change of variables formula (1) is the same for a continuous bounded variation functions A: if F is convex, then

$$F(A_T) = F(A_0) + \int_0^T F_x^+(A_s)dA_s.$$

- A natural question: to what extend fractional Brownian motion behaves as a continuous function with bounded variation?
- Representation of the running maximum $A_t^* = \max_{s\leq t} A_s$ of a continuous bounded variation function A with $A_0 = 0$:

$$A_t^* = \int_0^t 1_{\{A_s^* = A_s\}} dA_s; \tag{2}$$

this is a result by Azmoodeh, Tikanmäki, V. [work in progress].

Main results

Running maximum of a continuous bounded variation function A

The change of variables formula (1) is the same for a continuous bounded variation functions A: if F is convex, then

$$F(A_T) = F(A_0) + \int_0^T F_x^+(A_s) dA_s.$$

- A natural question: to what extend fractional Brownian motion behaves as a continuous function with bounded variation?
- Representation of the running maximum $A_t^* = \max_{s \leq t} A_s$ of a continuous bounded variation function A with $A_0 = 0$:

$$A_t^* = \int_0^t 1_{\{A_s^* = A_s\}} dA_s; \qquad (2)$$

  this is a result by Azmoodeh, Tikanmäki, V. [work in progress].
- Apparently (2) does not generalize to fractional Brownian motion.

Speculations

Consider the process X^ϵ, where $X_t^\epsilon = B_t^H + \epsilon W_t$; here the Brownian motion W is independent of the fractional Brownian motion of B^H, $H > \frac{1}{2}$.
Assume now that the random variable $F = F(W_T, \eta^1, \eta^2, \ldots, \eta^k) \in F_T^W$ has an integral representation

$$F = c + \int_0^T f(W_s, \eta_s^1, \ldots, \eta_s^k) dW_s,$$

where η^j are so-called hindsight functionals like W_s^*, $\int_0^s W_u du$ etc.

Speculations

Consider the process X^ϵ, where $X_t^\epsilon = B_t^H + \epsilon W_t$; here the Brownian motion W is independent of the fractional Brownian motion of B^H, $H > \frac{1}{2}$.
Assume now that the random variable $F = F(W_T, \eta^1, \eta^2, \ldots, \eta^k) \in F_T^W$ has an integral representation

$$F = c + \int_0^T f(W_s, \eta_s^1, \ldots, \eta_s^k) dW_s,$$

where η^j are so-called hindsight functionals like W_s^*, $\int_0^s W_u du$ etc.
It follows from recent work by Bender, Sottinen, V. that the functional F^ϵ with respect to the paths of X^ϵ has the same integral representation :

$$F^\epsilon = c + \int_0^T f(X_s^\epsilon, \eta_s^1(\epsilon), \ldots, \eta_s^k(\epsilon)) dX_s^\epsilon.$$

Speculations and closing

If one now lets $\epsilon \to 0$ in

$$F^{\epsilon} = c + \int_0^T f(X_s^{\epsilon}, \eta_s^1(\epsilon), \ldots, \eta_s^k(\epsilon)) dX_s^{\epsilon}.$$

,
when it is true that one gets

$$F^H = c + \int_0^T f(B_s^H, \eta_s^1(H), \ldots, \eta_s^k(H)) dB_s^H ?$$

Speculations and closing

If one now lets $\epsilon \rightarrow 0$ in

$$F^{\epsilon} = c + \int_0^T f(X_s^{\epsilon}, \eta_s^1(\epsilon), \ldots, \eta_s^k(\epsilon)) dX_s^{\epsilon}.$$

,
when it is true that one gets

$$F^H = c + \int_0^T f(B_s^H, \eta_s^1(H), \ldots, \eta_s^k(H)) dB_s^H?$$

I hope that before my next talk on this topic we will have some new results on this.

Speculations and closing

If one now lets $\epsilon \to 0$ in

$$F^{\epsilon} = c + \int_0^T f(X_s^{\epsilon}, \eta_s^1(\epsilon), \dots, \eta_s^k(\epsilon)) dX_s^{\epsilon}.$$

,

when it is true that one gets

$$F^H = c + \int_0^T f(B_s^H, \eta_s^1(H), \dots, \eta_s^k(H)) dB_s^H?$$

I hope that before my next talk on this topic we will have some new results on this.

Thank you!

Aalto University